Jacques MARTINET

Updated March 17th, 2013;
references added on December 20th, 2017

Erratum to

Lenstra's constant and Euclidean number fields
by Armin LEUTBECHER and Jacques MARTINET,
Astérisque **94** (1982), 87–131.

I intend to return to this paper in order to try to improve the upper bounds for Lenstra's constant $M(K)$, in particular those given in Sections 4,5,6 for fields K of unit rank 2, and also to consider the parity of narrow class numbers, as Theorem 4.3.1 and the remarks which follow, but leaving aside the applications to Euclidean fields.

While revisiting this paper I found a counter-example to the result "proved" in Proposition 4.1.2: it may happen that *none of the five orbits listed in line 4 of the statement* contains an exceptional unit of the form $\pm\lambda^2$; see *Erratum* below.

Erratum.
Proposition 4.1.2. The error (for which I am the only responsible) occurs in the last three lines of the proof: the units we exhibit are indeed of the form $\pm\lambda^2$, but *they are not exceptional*!

Actually the error could have been found using data displayed in Subsection 5.2: there are exactly 14 fields $[K_1$—$K_{14}]$ containing exceptional units which are a square and in Remark 5.2.2, two fields, say, K_{15} and K_{16}, which do not belong to the previous list, are quoted, which have $M = L = 5$, and for which one can easily check that the five orbits (actually, only four, since v belongs to the orbit of u) do not contain any exceptional unit of the form $\pm\lambda^2$, having discriminants 3889 and 4417, respectively:

$$K_{15} \colon f_{15} = X^5 - X^4 - X^2 + X - 1\,;\quad K_{16} \colon f_{16} = X^5 + X^3 - 2X^2 + 2X - 1\,.$$

We have $M \geq 5$ thanks to the conditions "$0, 1, \zeta_6 \in U(f)$" (see Proposition 1.3.2), but none of the 24 exceptional units related to the sequence $0, 1, x, \frac{1}{1-x}$ (x a root of f_{15} or f_{16}) is the opposite of a square.
[Probably these two fields contain only 24 exceptional units.]

This error has no consequence on the results proved for fields with unit rank 2 and degree $4, 5, 6$, since the proofs only rely on Proposition 2.4.1 (and on Lemma 2.4.2 for degrees 4 and 5). However the proof for degree 3 is no longer valid, and the method used to handle degrees 4 and 5 only gives $M \leq 5$ for cubic fields. On *March 1st, 2013*, I wrote "I am pretty sure that the correct bound is 3 for $d_K > 49$, but I can only prove the following weaker result ... (indeed, $M \leq 4$ if $d_K > 49$). This is now a theorem.

Theorem. (Written on March 17th, 2013) *Let K be a totally real cubic field. Then we have $M = 7$ if $d_K = 49$ and $M = 2$ or 3 if $d_K > 49$.*

Proof. We know that the orbit under G of any exceptional unit contains a unit x which is a root of a polynomial

$$P_1 = X^3 - TX^2 + (T-3)X + 1 \text{ or } P_2 = X^3 - TX^2 + (T+1)X - 1$$

(where T is the *trace* $\mathrm{Tr}_{K/\mathbb{Q}}(x)$ of x), and that the discriminants of these polynomials are

$$d_1 = (T^2 - 3T + 9)^2 \text{ and } d_2 = (T^2 - 3T - 1)^2 - 32\,.$$

Moreover, we may assume (replacing x by $1-x$ if need be) that $T \geq 2$, and we then know that d_2 takes the values $-23, -31, -23, +49$ for $T = 2, 3, 4, 5$ and is a positive non-square if $T \geq 6$.

Assume now that K contains an exceptional sequence $0, 1, x, y$ of length 4. Observe that at least one of the exceptional units $x, y, \frac{y}{x}$ has norm $+1$, thus *is not* a root of a polynomial P_1. Hence if K is cyclic, we must have $d_K = 49$.

From now on we assume that x is a root of a polynomial P_2 with $T \geq 6$. Recall that no exceptional unit in K is a square (only the fields of discriminant -23 and 49 contain such units). The unit x is totally positive and $1 - x$ is negative at two real places. This defines a canonical real place v_1 such that all units of norm $+1$ are positive at v_1 and have the same sign at the other two real places v_2, v_3. (Such a canonical place exists whenever $h_K^+/h_K = 2$.)

Let $E_K^{v_1}$ be the group of units which are positive at v_1. By Lemma 2.4.2, if $M(K) \geq 4$, there exists an exceptional sequence $0, 1, x, y$ with both $v_1(x), v_1(y) \in (0,1)$. Since $\mathrm{Tr}(x) \geq 6$, x is totally positive. Since no exceptional unit of K is a square, $1, x, y, y/x$ are representatives for $E_K^{v_1}/E_K^2$. If y were totally positive, then y or $\frac{y}{x}$ would be a square. Hence y has signature $+--$.

Let $T = \mathrm{Tr}(x)$ and $T' = \mathrm{Tr}(1-y)$. Exchanging if necessary $0, 1, x, y$ and $0, 1, 1-y, 1-x$ we may assume that $T' > T$. For the polynomial P_2 with root x, we have $P_2(0) = -1$, $P_2(1) = 1$, $P_2(2) = 9 - 2T < 0$, $P_2(T-2) = -T^2 + 7T - 11 < 0$ and $P_2(T-1) = 2T - 3 > 0$, so that we can define the three real embeddings of K by

$$0 < v_1(x) < 1 < v_2(x) < 2 < T - 2 < v_3(x) < T - 1\,.$$

We also have $P_2(\frac{1}{T}) = \frac{1}{T^3} > 0$ and $P_2(\frac{1}{T+1}) = -\frac{T^2+T-1}{(T+1)^3} < 0$, which shows the more precise result $\frac{1}{T+1} < v_1(x) < \frac{1}{T}$. Similarly we have $\frac{1}{T'+1} < v_1(1-y) < \frac{1}{T'}$, thus $v_1(y) > 1 - \frac{1}{T'}$ and consequently $v_1(y) - v_1(x) > 1 - \frac{1}{T} - \frac{1}{T'} > 1 - \frac{2}{T}$. Since $v_2(y)$ and $v_3(y) < 0$, we have the lower bounds $v_2(x) - v_2(y) > 1$ and $v_3(x) - v_3(y) > T - 2$, hence

$$|N(y-x)| = \prod_{i=1}^{3} |v_i(y) - v_i(x)| > (1 - \tfrac{2}{T}) \cdot (T-2) \geq \tfrac{8}{3} > 1\,,$$

which contradicts the fact that $y - x$ is a unit. $\square$

More on cubic fields.

Experimentation suggests that totally real, non-Galois cubic fields contain at most one orbit of exceptional units. Cyclic fields of discriminants 49, 81, 169 and 361 — the smallest four discriminants of cyclic fields — are known to contain more than one orbit of exceptional units (7 for $d = 49$ and 3 for $d = 81$[1], probably 2 in the remaining two cases[2]). These four fields (together with that of discriminant -23) seem to be the only cubic fields containing more than six exceptional units.

Update of the reference list.

[14] J. Martinet, Petits discriminants des corps de nombres, London Math. Soc. Lectures Notes **56** (1982), 151–193.

[Lt] A. Leutbecher, Euclidean fields having a large Lenstra constant, Ann. Inst. Fourier **35** (1985), 83–106. [Announced in "Added in proof", p. 118.]

In [Lt], the polynomial given on the first line of p. 93 is not correct (it is reducible); a polynomial with the stated discriminant is

$$X^4 - X^2 - (\theta+1)X - \theta \quad (\theta^2 - \theta - 1 = 0)$$

which I constructed using Simon's list in

D. Simon, https://simond.users.lmno.cnrs.fr/ .

Numerous Euclidean fields in new signatures have been found by Julien Houriet by constructing long exceptional sequences for fields taken from Simon's list; see

[Hou] J. Houriet, Exceptional units and Euclidean number fields, Arch. Math **88** (2007), 425–433.

[1] T. Nagell, Sur un type particulier d'unités algébriques, Ark. Mat. **8** (1969), 163–184

[2] see V. Ennola, Cubic number fields with exceptional units, in Computational number theory (Debrecen, 1989); de Gruyter, Berlin (1991), 103–128