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TABLES for PART 3 of my HOMEPAGE.

We present here various tables of lattices, explaining first data that are common to all tables, then specific properties, one section per table. Actually, lattices are supposed to lie in some Euclidean space E and are represented by a Gram matrix in PARI-GP format. We denote by $S = S(L)$ the set of minimal vectors of a lattice L.

The following files from number **2** onward can be downloaded in a GP-session in order to get the Gram matrices; however the other data (lines beginning with \\) must be read using an editor (emacs,vi, ...).

The same lattice (up to similarity) may occur in various tables. Different Gram matrices for the same lattice have different names. Coincidences with a previous table are indicated.

All lattices listed here are *integral* (scalar products take only integral values on these lattices); they are also *primitive* (the scalar products have gcd = 1), except lattices Ln with $n \le 8$ in the file Lambda.gp.

Content.

3.1. Description of data relative to Gram matrices.

3.1a. Characteristic. This is a 4-component PARI-vector of the form

$$[d, [s, m], [s^*, m^*], [a_1^{k_1}.a_2^{k_2} \dots .a_p^{k_p}]]$$

where d is the determinant, $[s, m]$ the number of pairs of minimal vectors and their norm, $[s^*, m^*]$ the same data for the dual rescaled to a primitive form, and the last component is the *Smith invariant* of a lattice L represented by the given Gram matrix, i.e., the elementary divisors of the pair (L^*, L) (we have $L^* \supset L$ since L is integral).

Note that $\sum k_i = \dim L$ and $\prod a_i^{k_i} = \det(L)$, that a_1 is the annihilator of $:^* /L$, that $\min L^* = \frac{m}{a_1}$, and that $a_p = 1$ if L is primitive.

With these data we can calculate the *Hermite invariant* $\gamma(L)$ *of* L:

$$\gamma(L) = \frac{m}{d^{1/n}}\,,$$

and its dual analogue ("Bergé-Martinet invariant")

$$\gamma'(L) = \big(\gamma(l)\gamma(L^*)\big)^{1/2} = \frac{m\,m^*}{a_1}\,.$$

We complete this information by the order of the automorphism group of L and its prime decomposition, in the form "$|\operatorname{Aut}| =$".

3.1b. Data relative to extremality. A lattice L is *extreme* if γ attains a local maximum on L. If the Gram matrix we consider defines an extreme lattice, the mention EXTREME is written, except in a few cases when the stronger property of *strong perfection* to be defined later in **1.c** holds.

We now consider weaker properties. The *perfection rank* $\operatorname{perf} L$ *of* L is the rank in $\operatorname{End}^s(E)$ of the orthogonal projections to the lines containing the minimal vectors. We have $1 \le \operatorname{perf} L \le \frac{n(n+1)}{2}$. A lattice L with $\operatorname{perf} L = \frac{n(n+1)}{2}$ is *perfect*.

A lattice is *weakly eutactic* if the identity of E belongs to the span of the projections onto $S(L)$, i.e., if there is a representation

$$\operatorname{Id} = \sum_{x \in S/\pm} \lambda_x\, p_x$$

with real coefficients λ_x. (p_x is the orthogonal projection onto $\mathbb{R}x$.) It is *semi-eutactic* (resp. *eutactic*) if there exists such a representation with non-negative (resp. strictly positive) coefficients λ_x. By a theorem of Voronoi,

$$extreme \iff perfect \text{ and } eutactic\,.$$

The perfection rank of L is written "perf =" if L is not perfect. Otherwise we write the mention PERFECT. However, *unless otherwise stated,* PERFECT *is to be understood as* NOT EXTREME.

Weak eutaxy, semi-eutaxy, eutaxy are written WEUT, SMEUT, EUT, respectively. It is to be understood that WEUT implies NOT SMEUT nor PERFECT, that SMEUT implies NOT EUT, and EUT implies NOT EXTREME.

3.1c. Spherical designs. Spherical t-designs are special finite configurations on the sphere $\mathbb{S}^n$. A (spherical) t-design is a t'-design for any $t' \le t$, a 1-design is simply a symmetric set with respect to the origin, and a symmetric t-design with even t is a $t+1$-design.

Venkov has applied the theory of spherical designs to sets $S(L)$. In dimensions $n \ge 2$, $S(L)$ is t-design for a largest possible value of t, the *level* of S (or of L), necessarily odd. In the known examples, the level is one of $1, 3, 5, 7, 11$; I conjecture that this is general.

Lattices with level ≥ 3 are those which possess a eutaxy relation with equal coefficients λ_x. They were called *strongly eutactic* by Venkov. They are mentioned as STREUT; then, EUT implies NOT STREUT.

Lattices with level ≥ 5 were called *strongly perfect* by Venkov. They are mentioned as STRONGLY PERFECT; they are in particular extreme ans strongly eutactic. Thus we then do not mention EXTREME nor STREUT.

Consider a semi-eutactic, non-eutactic lattice and suppose that there is a partition $S = S_0 \cup S_1$ so that eutaxy coefficients are 0 on S_0 and equal on S_1. Then S_1 is a spherical 3-design. Lattices having this property are called *strongly semi-eutactic*. This property is also mentioned.

3.1d. Some more data. Less systematically than above we mention some other properties: transitivity properties of $\mathrm{Aut}(L)$ on S and S^*, even the rank in case of a transitive action, or also properties relative to γ' (DUAL-EXTREME, DUAL-EUT(actic). We scarcely mentioned these latter properties: indeed, if L is extreme and L^* is at least semi-eutactic, L and L^* are dual-extreme; in the other direction, if $\mathrm{rk}\, S < n$ or $\mathrm{rk}\, S^* < n$, L and L^* are not dual-extreme.

Also modular lattices are quoted. (m-*modular* means that there is a similarity of module m which maps L^* onto L; 1-*modular* is equivalent to *unimodular*).

3.1e. Laminations Due to their importance in the constructions of the tables, we briefly describe laminations, then antilaminations.

Start with a lattice L_0, then consider representatives $L_{1a}, L_{2a}, \ldots$ of the set of isometry classes of lattices containing L_0 as a hyperplane section, with the same minimum as L_0, and of determinant as small as possible (i.e., Hermite invariant as large as possible); then perform the same construction above L_{1a}, L_{1b}, etc. These are the *weak laminations over* L_0.

From the second step onward, it may happen that different determinants occur. Keeping only lattices having the smallest determinant, we obtain the (*strong*) *laminations over* L_0.
Basic example: Conway-Sloane's *laminated lattices*, where we start with the lattice $\{0\}$ to which we give minimum 4 (for convenience). There is only one *cul-de-sac* (named $\Lambda_{13}^{\mathrm{mid}}$, L13mid in PARI), and the Leech lattice is the unique laminated lattice in dimension 24.

A third type of laminations are Plesken-Pohst's *integral laminations*. This time we start with an integral lattice L_0 and consider integral lattices $L_{1a}, L_{2a}, \ldots$ having the same minimum as L_0 and of determinant as small as possible; then go on above L_{1a}, L_{1b}, etc.

If one obtains at some step a unimodular lattice U, then the series continues with $U \perp L_{1a}$,... It is easily seen that this is the case with $L_0 = \mathbb{Z}$ ($U = \mathbb{Z}$) and $L_0 = \mathbb{A}_1$ ($U = E_8$). Plesken and Pohst have shown that starting with scaled copies of $\mathbb{Z}$ to minimum 3 (resp. 4), we meet the unique unimodular lattice of minimum 3 (resp. 4) and dimension 23 (resp. 24, the Leech lattice). What happens with larger minima is not known.

3.1f. Antilaminations. Whereas laminations are constructed using ascending constructions, antilaminations are constructed using descending constructions.

Start with a lattice L_0 and consider its hyperplane sections of minimal determinant. These are in one-to-one correspondence with the minimal vectors of L_0^*. In practice, we list representatives of orbits in $S(L_0^*)$, determine their orthogonal in L_0, finally remove redundancies if need be, then go on, starting with the above list. This process increases the minima. However we often find long chains of lattices with the same minima as L_0.

We compare below antilaminations and various laminations on two examples.

The Leech lattice Λ_{24}. In dimensions 14–24, (strong) laminations and antilaminations produce the same, unique lattices. In dimensions $n = 13$, 12, 11, there are three, three, two laminated lattices, respectively. Antilaminations produce only two lattices if $n = 13$ ($\Lambda_{13}^{\mathrm{mid}}$ is missing), next the same three lattices for $n = 12$, and one more lattice if $n = 11$ (denoted by $\Lambda_{11}^{\mathrm{mid}}$; *not* a laminated lattice), a cul-de-sac if we restrict ourselves to *strong* antilaminations.

The shorter Leech lattice O_{23}. Antilaminations of O_{23}, the unique unimodular lattice of minimum 3 in dimension 23 produce a unique lattice O_n for $23 \geq n \geq 14$, among which the strongly perfect lattices O_{22} and O_{16}. Arithmetic laminations of Plesken and Pohst produce six lattices with $n = 15$, including the extreme $plp15f$, all distinct from O_{15}, then six lattices with $n = 16$, again all distinct from O_{16}. Thus antilaminations (resp. arithmetic laminations) do not find the extreme lattice $plp15f$ (resp. O_{16}).

3.2. Lambda.gp

The file contains 6 parts: (1) laminated lattices (and some "companion" lattices); (2) Plesken-Pohst laminations for minimum 4 (when not in part 1); (3) some antilaminations of "L11mid" (the orthogonal of L13mid in L24, *NOT a laminatef lattice*); (4) the series $K_n = Kn$;

(5) the series $K'_n = Kpn$; (6) varia (especially a few lattices with a somewhat large invariant γ' with respect to more "classical" lattices).

3.3. AntilaminO$_{23}$.gp

The *shorter* (resp. *odd*) *Leech lattice* is the unique unimodular lattice of minimum 3 and dimension 23 (resp. 24), denoted by O_{23} =OO23 (resp. O_{24} =OO24. Part 4 lists a few antilaminations of OO24, whereas Parts 1,2,3,5 concern sublattices of OO23.

The smallest three determinants of 2-dimensional sublattices of ${\mathrm{O}_{23}}^* \simeq \mathrm{O}_{23}$, are $8, 9, 11$. They constitute a unique orbit, represented my matrices $[3,1;1,3]$, $[3,0;0,3]$, $[3,1;1,4]$, respectively, Orthogonal lattices in O_{23} are characterized by their even sublattices, L21 and Kp21 in Lambda.gp for the first two. [Then there are two orbits on determinant-12 matrices, represented by $[3,0;0,4]$ and $[4,2;2,4]$, respectively; their antilaminations have not been computed.]

(1) deals with the antilaminations of OO23 in dimensions $23 \ge n \ge 8$, denoted by OOn, with an extra subscript when they are not unique. (The orthogonal to $[3,1;1,3]$ is OO21.)
(2) deals with the antilaminations of $Op21$, the orthogonal to a matrix $[3,0;0,3]$ in OO23*.
(3) deals with the antilaminations of $Oq22$, the orthogonal of a norm-4 vector in OO23*; the next step Oq21 is the orthogonal in OO23 of the matrix $[3,1;1,4]$.
(4) deals with the antilaminations of OO24 (denoted by Zn?) in dimensions 24 to 22.
(5) contains some more lattices of minimum 3.

3.4. Min3.gp

The file contains three parts.

(1) Plesken-Pohst arithmetic laminations.

(2) Lattices of Minimum 3 from Minimum 4. Integral lattices of minimum 3 are constructed of the form $L = L_0 \cup (L_0 + \frac{e}{2})$ with L_0 of minimum 4 and $e \in L_0$ of norm 12.

(3) Some More Sections of O_n for Large n.

3.5. Min5.gp

The first three part are devoted to integral lattices of minimum 5 generated by minimal vectors of pairwise scalar products ± 1. Their minimal vectors support equiangular families of lines, and moreover, they are sometimes useful to construct strongly regular graphs. These are uniquely defined, antilaminations of three lattices Qa14, Qb15, Qc23.

(Qa23 is strongly perfect; Qa14 is an extension of its section Qa13, $C2 \times PSL(2,25) : C2$ in Nebe-Sloane's catalogue).

A fourth part in construction will be devoted to various other integral lattices of minimum 5.

3.6. Min6.gp

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Parts 1 to 4 are devoted to antilaminations of specific lattices of dimensions 10 ($Kp10^*$), 16 (a strongly perfect lattice of Hu and Nebe), 18 ($Kp18^*$), and 22 (a strongly perfect lattice related to Leech), respectively.

Various other lattices are displayed in Parts 5 to 8.

3.7. Minlarge.gp

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Part 1 contains one lattice ($M11_{22}$) of minimum 11 derived from $Qc22$, of minimum 5. Part 2 (resp. 3) contains lattices derived from $M11_{22}$ of minimum 10 (resp. 12). For lattices displayed in parts 1 and 3 the absolute value of scalar products of minimal vectors have only two values.

3.8. Modular.gp.

A lattice L is *m-modular* ($m \geq 1$ an integer) if there exists a a similarity of modulus m which maps L^* onto L. Rescaling to determinant 1, we obtain a rational, isodual lattice, and conversely, a rational, isodual lattice is proportional to a modular lattice.

We have reproduced only a few examples of 1-, 2- or 3-modular lattices, for which many classification results are known. Similarly, we have displayed only a few examples of lattices constructed using tensor products or exterior powers: $L_1 \otimes L_2$ for L_1, L_2 modular or $L_2 = L_1^*$; $\wedge^k L_0$ for L_0 modular or $\dim L_0 = 2k$.

The characteristics have been given a simplified form since data for L and L^* are the same.