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WEB PAGES ON MINIMAL CLASSES

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Abstract. We first recall the main properties of minimal classes, then explain the data which can be downloaded (file *Sdim2to4.gp*, then describe various complements (groups, domains, reduced domains, dual-minimal classes) which can be read with an editor, but not downloaded.

Introduction and Note on References

This text is a wide development of a former Sections 6 to 8 of that was previously "Voronoi graphs and minimal classes", now restricted to "Voronoi graphs". Enlargements concern Sections 3 to 6 below.

This files consists of the following sections:

1. *Minimal Classes and (Weak) Eutaxy.*
2. *The file* Sdim2to4.gp.
3. *Minimal Classes: Classification and Index Theory.*
4. *The Domains.*
5. *Reduction.*

In Construction:

6. *Dual-Minimal Classes.*
7. *Dual-Minimal Classes: Experimental Results.*

Note on references. I have kept the notation of the (enlarged) reference list of [M], except that I have replaced "[Mar]" by "[M]" to avoid a notational problem with the 1996 French edition of [M]. It is useful to complement [M] by references to be downloaded from [Mweb], in particular three references from Section I of [Mweb] (*Recent journal publications and preprints*), relative to [M]:

(1) [Merr] Erratum;
(2) [Mcpl] Complements;
(3) [Mbib] Corrected and Extended Reference List (update on pages 1 to 15, new references from page 16 onward).

Warning. Only a few references, directly related with this file, may be accessed to directly in the reference list at the end of this note; I refer to [Mbib] for the others.

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1. Minimal Classes and (Weak) Eutaxy

This text is a continuation of the text "Voronoi graphs" above, to which we refer the reader for some definitions which have not been reproduced below. We follow [M], Chapters 9 (for the theory) and 14 (for the numerical data, and [Bt] for the (not yet available here) data for dimension 5. Lattices are assumed to be contained in a given Euclidean space E, of dimension denoted by n. We denote by $s(\Lambda) = s$ the number of pairs $\pm x$ of minimal vectors of the lattice Λ.

1.1. **Basic definitions.** In terms of lattices, *minimal classes* are the classes for the equivalence relation on the set of lattices in a given n-dimensional Euclidean space:

$$\Lambda \sim \Lambda' \iff \exists u \in \mathrm{GL}(E),\, u(\Lambda) = \Lambda' \text{ and } u(S(\Lambda)) = S(\Lambda')\,.$$

We then define on these classes an ordering relation by

$$\mathcal{C}' \prec \mathcal{C} \iff \exists\, \Lambda \in \mathcal{C},\, \exists\, \Lambda' \in \mathcal{C}',\, S(\Lambda') \subset S(\Lambda)\,.$$

In terms of quadratic forms, we consider the finite subsets $S \in \mathbb{Z}^n$ which are such that the set $\mathcal{Q}(S)$ of positive definite quadratic forms q such that $S = S(q)$ is not empty. Then $\mathcal{Q}(S)$ is a convex open polyhedron. We obtain this way a cell decomposition for the set of positive definite quadratic forms. Minimal classes correspond to equivalence classes of cells under the action of $\mathrm{GL}_n(\mathbb{Z})$.

Recall that a lattice Λ (or a positive definite quadratic form) q is *well rounded* if it contains $n = \dim \Lambda$ independent minimal vectors, and that a class is *well rounded* if all its lattices are. In the sequel, we restrict ourselves to well-rounded classes. This is no important restriction: classes whose minimal vectors have rank $n' < n$ are in one-to-one correspondence with n'-dimensional classes.

The perfection rank r (and the cardinality s) of S are invariants of the class $\mathcal{C}$ defined by S. The dimension of a cell is its *perfection co-rank*, namely $\frac{n(n+1)}{2} - r$. The 0-cells are perfect forms and 1-cells are Voronoi paths connecting two perfect forms. Thus minimal classes are objects which generalize both perfect forms (or lattices) and Voronoi paths. They are related to a construction we considered in Section 2 of *Voronoi graphs*: given S, $S' \subset S$ defines a class $\mathcal{C}'$ (with necessarily $\mathcal{C}' \prec \mathcal{C}$) if and only if it is admissible in the sense of Definition 2.1 of *Voronoi graphs* and satisfies moreover sign conditions with respect to faces. The topological closure $\overline{\mathcal{C}}$ of a class $\mathcal{C}$ is the union of the classes $\mathcal{D} \succ \mathcal{C}$ (we say that $\mathcal{D}$ lies above $\mathcal{C}$).

1.2. **Characterization of Minimal Classes and automorphisms.** A minimal class $\mathcal{C}$ is well defined by the $n \times s$ matrix of the components of the set S of the minimal vectors up to sign of some lattice $\Lambda \in \mathcal{C}$ on a basis $\mathcal{B}$ of Λ over $\mathbb{Z}$. The problem arises of comparing two classes $\mathcal{C}_1, \mathcal{C}_2$ given by sets S_1, S_2, respectively. To this end we attach to every S the $n \times n$ matrix $M_S := S\,{}^tS$. This is a matrix with entries in $\mathbb{Z}$, and since we restrict ourselves to well-rounded classes, M_S is positive definite.

Theorem 1.1. *The class of M_S modulo $\mathrm{GL}_n(\mathbb{Z})$ (or the isometry class of a lattice with Gram matrix M_S) does not depend on the choice of S, and the assignment $\mathcal{C} \to \mathrm{cl}(M_S)$ is injective.*

[We thus attach to every class $\mathcal{C}$ the various invariants of the isometry class of lattices with Gram matrix M_S. Except for a few of them (see Remark 1.3) I do not know whether they have a sensible interpretation in terms of minimal classes.]

The isomorphism of a class $\mathcal{C}$ onto itself is the *automorphism group* $\mathrm{Aut}(\mathcal{C})$ *of* $\mathcal{C}$, that we shall generally denote by G; the subgroup of $\mathrm{Aut}(\mathcal{C})$ of those automorphisms which stabilize *all* lattices in $\mathcal{C}$ is the *strict automorphism group* $\mathrm{Aut}(\mathcal{C})_0$ *of* $\mathcal{C}$, generally denoted here by G_0.

The group G will play a major rôle in the question of reduction that we shall consider later, and the group G_0 is useful in questions related to duality. Note that if a group H acts on a lattice Λ, then the corresponding action of $u \in H$ on the dual lattice Λ^* is given bu $u * x = {}^tu^{-1}(x)$. Thus if Λ is acted on by G, then Λ^* is acted on by the transpose of G. Note that

the group G is *canonically isomorphic to* $\mathrm{Aut}(M_S^{-1})$

(A.-M. Bergé; see [Bt], Prop. 2.9).

Also, for every $A \in \mathcal{C}$, the corresponding *barycenter matrix*

$$M = \tfrac{1}{|\mathrm{Aut}(S)|} \textstyle\sum_{U \in \mathrm{Aut}(S)} {}^tU A\, U$$

has the full group $\mathrm{Aut}(\mathcal{C})$ as its automorphism group.

1.3. **Eutaxy.** For $x \in E \smallsetminus \{0\}$, denote by $p_x \in \mathrm{End}^s(E)$ the orthogonal projection to $\mathbb{R}x$. Given a lattice Λ (resp. a positive definite quadratic form q with matrix A), a *eutaxy relation for* Λ (resp. *for* q) is an equality

$$\mathrm{Id} = \sum_{x \in S(\Lambda)} \rho_x\, p_x \quad \text{(resp. } A^{-1} = \sum_{X \in S(q)} \rho'_X\, X\,{}^tX) \,.$$

[These definitions are compatible with the dictionary *lattices* $\longleftrightarrow$ *quadratic forms*: if $\mathcal{B}$ is a basis for Λ with Gram matrix A, then $A = \mathrm{Mat}(Id, \mathcal{B}, \mathcal{B}^*)$ and $X^t X = \mathrm{Mat}(p_x, \mathcal{B}^*, \mathcal{B})$ — note the exchange $\mathcal{B} \longleftrightarrow \mathcal{B}^*$.]

Definition 1.2. We say that a lattice or a form is *weakly eutactic* if there exists a eutaxy relation between its minimal vectors. We say that it is *semi-eutactic* (resp. *eutactic*) if moreover the eutaxy coefficients can be chosen to be non-negative (resp. strictly positive). We say that it is *strongly eutactic* if there exists a eutaxy relation with equal coefficients. (It is also useful to consider *strongly semi-eutactic* lattices and forms, those for which there exists a eutaxy relation with equal non-zero eutaxy coefficients.)
[Strongly eutactic lattices are the lattices whose set of minimal vectors is a spherical 2-design (or 3-design, this amounts to the same by central symmetry). Similarly, strongly semi-eutactic lattices are the lattices whose set of minimal vectors having *non-zero* eutaxy coefficients is a spherical 3-design.]

On the closure of a given class $\mathcal{C}$, the Hermite invariant γ (defined by $\gamma(\Lambda) = \dfrac{\min \Lambda}{\det(\Lambda)^{1/n}}$) attains a minimum. By a theorem of A.-M. Bergé and J. Martinet, this minimum is attained on a unique lattice (up to similarity), which is also the unique weakly eutactic lattice (up to similarity) in its class. This class is some class $\mathcal{C}' \subset \overline{\mathcal{C}}$ (whence a canonical map $\mathcal{C} \mapsto \mathcal{C}' \succ \mathcal{C}$).

Remark 1.3. A minimal class $\mathcal{C}$ contains a strongly eutactic lattice if and only if the matrix M_S^{-1} is strongly eutactic, and M_S^{-1} is then *the* weakly eutactic lattice of $\mathcal{C}$.

2. The file *Sdim2to4.gp*

This file, devoted to minimal classes in dimensions $n = 2, 3, 4$, can be downloaded in PARI-GP. It consists of four parts, the first three of which contain data which can be downloaded; by editing this file (under *emacs, vi*, etc) one can read complementary information.

Since we consider only dimensions $n \le 4$, the lattices of every well-rounded minimal class have a basis of minimal vectors (and even, to within the unique exception of the perfect class of the lattice $\mathbb{D}_4$, any set of n independent minimal vectors is a basis. (In $\mathbb{D}_4$, such a set generates a sublattice of index 1 or 2.)

Here is a more precise description.

Part 1: The Classes. This part contains a list of systems S of components of minimal vectors (up to sign) on a basis of minimal vectors, following the notation of [M], Chapter 9, chosen so that the first non-zero component be positive. The set of minimal vectors is written in the form $e_1, \dots, e_n, e_{n+1}, \dots, e_s$. Consequently the matrices for S have the form $S = (I_n \mid S_1)$ where $S1$ is an $n \times (s-n)$ matrix.

The notation is Snxr, where n is the dimension, $r \in [n, \frac{n(n+1}{2}]$ is the perfection rank, and x is one of the letters a, b, c, d.

Editing the file one can read the orders of the corresponding automorphism groups.

Part 2: Gram Matrices. These matrices are $n \times n$ matrices scaled to minimum 2, hence with diagonal entries 2, depending on k parameters $x_1, \dots, x_k$ where k is the perfection co-rank of the class. Thus we have

$$k = \frac{n(n+1)}{2} - r\,, \quad \text{hence} \quad 0 \le k \le \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}\,.$$

The entries of these matrices are affine functions of the x_i. They define a lattice of minimum 2 provided we restrict ourselves to a domain defined by affine inequalities; see Section 4 below.

Part 3: Eutaxy. The $2+5+18 = 25$ minimal classes in dimensions 2 to 4, except $S4a6$, contain a weakly eutactic lattice, and these lattices are eutactic except that of $S4b7$, which is only semi-eutactic, and are rational except those of $S4a7$ and $S4a8$, the fields of definition of which are quadratic.

Part 3 contains Gram matrices for these lattices, denoted by $Mnxr$ ("M..." instead of "S..." used for the class); $M4a7$ and $M4a8$ are given in the PARI-format for algebraic numbers.

Part 4: Strict Automorphisms. Editing the file one can read the orders of automorphism groups given in part 1 together with the minimal vectors of the class; in the short part 4, which can only be read, we display the orders of the *strict automorphism group of each class*, i.e., the subgroup G_0 of $G := \mathrm{Aut}(\mathcal{C})$ which stabilizes all lattices in $\mathcal{C}$.

3. Minimal Classes: Classification and Index Theory

The classification of minimal classes is known up to dimension $n = 7$, and can be read in [M], Chapter 9 for $n \le 4$, where this is established inductively on the perfection rank (result due to Štogrin, completed by Bergé and Martinet; see [St] and [B-M5] of [Mbib]).
[The results for $n = 5$ can be read in [Bt]; for $n = 6, 7$, one can contact the authors of [E-G-S2].]

3.1. **Index Theory.** We now turn to index theory. Recall that the lattices we consider are assumed to be well-rounded (otherwise we must consider the notion of a *Minkowskian sublattice*; see [K-M-S]).

Let $x_1, \dots, x_n$ be n independent minimal vectors in a lattice Λ, and let Λ' be the sublattice of Λ they generate (thus, $(x_1, \dots, x_n)$ is a basis for Λ'). The index $[\Lambda : \Lambda']$ is bounded from above by $\gamma(\Lambda)^{n/2}$. The set of possible isomorphism classes of quotients Λ/Λ' (and the set of possible values for $[\Lambda : \Lambda']$) are invariants of the class of Λ. If $n \le 4$, we have $\gamma(\Lambda)^2 \le 2$ with equality only on the class of $\mathbb{D}_4$. Thus this index is equal to 1 except on $\mathrm{cl}(\mathbb{D}_4)$ where it is equal to 1 or 2, and in all cases, lattices have a basis of minimal vectors.
[In particular they are generated by their minimal vectors. The converse is true up to dimension 9, but not beyond dimension 9; cf. [Mar-Schr1].]

3.2. **Characteristic Determinants.** We restrict ourselves to lattices having a basis $\mathcal{B} = (e_1, \dots, e_n)$ of minimal vectors.

Let $k \le n$, and consider k minimal vectors $x_i = a_{1,i}e_1 + \cdots + a_{n,i}e_n$ distinct from the basis vectors. Following Korkine and Zolotareff ([K-Z3]; see [M], Sections 6.1 and 6.4), the absolute value of the determinant of a $k \times k$ matrix extracted from the components $a_{j,i}$ of the x_i is called a *characteristic determinant*. To each characteristic determinant equal to some $d \ne 0$ corresponds a lattice contained to index d in Λ. Characteristic determinants of order 1 are the $|a_{j,i}|$.

If the $a_{j,i}$ are all 0 or ± 1, then characteristic determinants of order 2 are either 0, ± 1 or come from pairs of components of the form $\{(1,1),(1,-1)\}$ (up to permutation and change of signs of the x_i). In the same conditions, if characteristic determinants of order 1 or 2 are equal to 0 or 1, then characteristic determinants of order 2 are either 0, ± 1 or come from sets $(1,1,0),(1,0,1),(0,1,1)$. For $n \le 4$, computing directly characteristic determinants of order 4 (which do not reduce to a single type), we obtain efficient restrictions on the possible sets of minimal vectors.

4. The Domains

With a minimal class $\mathcal{C}$ defined by a set S of components of minimal vectors on a basis we attached a matrix M depending affinely on k generators x_i (k is the perfection co-rank), but these parameters are not arbitrary. The *domain* $\mathcal{D}$ *of* $\mathcal{C}$ is the (open) subset of $\mathbb{R}^k$ to which the x_i must belong in order that M be a matrix of minimum 2 representing lattices in $\mathcal{C}$. The closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ is a closed convex polytope, which is the convex hull of its extremal points that we shall rather describe as an intersection of affine half-spaces, and $\mathcal{D}$ is the interior of $\overline{\mathcal{D}}$.

In what follows we shall discard the case of perfect minimal classes, for which there are no parameters (they correspond to polytopes reduced to a single point).

The domain can be characterized by the inequalities $N(x)- \geq 0$ for x running through all vectors $a_1e_1+\cdots+a_ne_n$. This produces infinitely many inequalities, but the facets of $\mathcal{D}$ correspond to the case when x may be minimal, and using of characteristic determinants we reduce ourselves to finitely many inequalities (a well known fact in the theory of reduction of quadratic forms).

Even when this is done there may exist redundancies. For instance using vectors $x = e_1 \pm e_2$ implies inequalities of the form $u(x_1, \dots, x_k) \geq c$ for some linear form u and $c \in \mathbb{R}$, which are empty if one of the vectors $e_1 \pm e_2$ is minimal. Other example: it happens that we find sets of inequalities of the form $-1 < x_1, x_2 < 1$ and $x_1 + x_2 < -1$. Then this implies $-1 < x_1, x_2 < 0$ and we may forget the upper bounds $x_1, x_2 < 1$.

The domains for $n \leq 4$ were obtained using these considerations. To each class $Snxr$ and each matrix $Mnxr$, the set of inequalities which define the domain are referred to $Dnxr$. The results will be displayed in the $future\ text-file$ *domains.txt in construction*; I have tried to limit redundancies. However the systems of inequalities displayed in this file are not claimed to be all minimal.

This file will also contain reduced domains, denoted by $Dnxr$, that we study in Section 5 below.

5. Reduction

As above we consider classes $\mathcal{C}$ defined by a set S and their corresponding Gram matrices M, and discard perfect classes. The matrices M depend on $k \geq 1$ parameters $x_1, \dots, x_k$, and their entries are affine functions of the x_i. The group $G := \mathrm{Aut}(\mathcal{C})$ is represented by matrices A_g, $g \in G$ and acts on the set of matrices M by $g \cdot M = {}^t A_g M A_g$.

Remark 5.1. Given a positive, definite matrix P with entries in $\mathbb{Z}$, the command *qfauto(P)* of *PARI* outputs a 2-component vector, the order of $G := \mathrm{GL}(P)$ and a vector with matrix components $G[i]$; as generators of $\mathrm{GL}(P)$ we may take the matrices $G_i := {}^t G[i]$.

The action of G on the set of matrices M preserves the constant entries. We may thus consider the action of G on the set $\mathcal{E}$ of non-constant entries. We denote by G_1 the *core* of the action. This is a subgroup of G of even order (because it contains $\{\pm \mathrm{Id}\}$), and which may be larger.

Example 5.2. Let $\mathcal{C} = S3a4$. This is the class of $\mathbb{A}_3^*$, so that we may identify $G/\{\pm \mathrm{Id}\}$ with $\mathfrak{S}_4$ acting on $\{e_1, e_2, e_3, e_4 := -(e_1 + e_2 + e_3)\}$. We have $\mathcal{E} = \{x_1, x_2, -2 - x_1 - x_2\}$, and $G/\{\pm \mathrm{Id}\} \simeq \mathfrak{S}_4$ acts on $\mathcal{E}$ through its quotient $\mathfrak{S}_3$.

The reduction procedure aims to write $\mathcal{D}$ as the union of translated under G of a sub-domain $\mathcal{R}$ so that any matrix in $\mathcal{D}$ be of the form $g \cdot M = {}^tA_g * M * A_g$ for some $g \in G$ and $M \in \mathcal{R}$, and that for any $g_1, g_2 \in G$, the dimension of $(g_1 \cdot M) \cap (g_2 \cdot M)$ be smaller than the dimension of $\mathcal{D}$.

Our reduction has been done using the following method.

If G acts trivially on $\mathcal{E}$, there is nothing to do. Otherwise pick $y_1 \in \mathcal{E}$ such that its orbit o_1 under G is not reduced to $\{y_1\}$ and consider the stabilizer $G^{(1)}$ of y_1. If $G^{(1)}$ does not act trivially on $\mathcal{E}$, pick $y_2 \in \mathcal{E}$such that $o_2 := G^{(1)}\, y_2$ is not reduced to y_2, then consider the action of the stabilizer $G^{(2)}$ of y_1 and y_2, etc. We finally obtain subsets $o_1, \dots, o_m$ and a descending chain of subgroups $G^{(1)} \supset \cdots \supset G^{(m)}$ such that $G^{(m)}$ acts trivially on $\mathcal{E}$. We then have $\prod_i |o_i| = |G/G_1|$, and have found G_1.

Now we may take as a reduced domain the subset $\mathcal{R}$ of $\mathcal{D}$ which defined by the inequalities $y_i \le \min o_i$ for $i = 1, \dots, m$. In other words, we must add these inequalities to the strict defining inequalities of $\mathcal{D}$. This procedure introduces new redundancies, that we have more or less suppressed, that the results displayed in the file *domains.txt* are not claimed to be optimal in this respect.

6. Dual-Minimal Classes **(Unfinished)**

Recall that the transpose ${}^tu$ of $u \in \mathrm{GL}(E)$ is defined by the condition

$$\forall\, x, y \in E, u(x) \cdot y = x \cdot {}^tu(y)\,,$$

so that for any lattice $\Lambda \subset E$, we have $u(\Lambda)^* = {}^tu^{-1}(\Lambda^*)$. The notion of a *dual-minimal class* consists in adding to the definition of a *minimal class* a condition involving duality.

Definition 6.1. We say that two pairs (Λ, Λ^*) and (Λ', Λ'^*) are *dual-minimal-equivalent* if there exists $u \in \mathrm{GL}(E)$ such that

$$u(\Lambda) = \Lambda' \;\; u(S(\Lambda)) = S(\Lambda') \;\text{ and }\; {}^tu^{-1}(S(\Lambda^*)) = S(\Lambda'^*)\,.$$

We define this way a partition of minimal classes into finitely many dual-minimal classes. We describe in this section what we know about these partitions for $n \le 4$. Actually our knowledge is very poor: these partitions are known only if $n = 2$, $n = 3$, and $n = 4$, $s \ge 7$. For the 8 remaining classes, conjectures and a few partial results will be given in Section 7.

Notation. The elements of the partition of a minimal class $\mathcal{C}$ are denoted using superscripts (1), (2), etc.. the value of s^*, and if there are several dual classes with, say, $s^* = 1$, we shall denote them using superscript $(1a)$, $(1b)$, etc. Thus we write $\mathcal{C}^{(1)}, ...,$ or $\mathcal{C}^{(1a)}, \mathcal{C}^{(1b)}, ...$

A future file *Partition.txt* will (?) display the partitions into dual-minimal classes (proved or conjectural) in dimensions $n \le 4$.

6.1. **Isodual lattices and classes.**

Definition 6.2. (1) We say that a lattice Λ is *isodual* if there exists an isometry σ (an *isoduality*) which maps Λ onto Λ^*. We say that Λ (or σ) is *symplectic* if $\sigma^2 = -\operatorname{Id}$.

(2) We say that a set of lattices $\mathcal{E}$ is (*weakly*) *isodual* if there exists a similarity σ which maps every lattice in $\mathcal{E}$ onto the dual of some lattice of $\mathcal{E}$, and that $\mathcal{C}$ is *strongly isodual* if σ may be chosen so as to map each lattice onto its dual.

[We shall often give (1) a weak sense, assuming only that σ is a similarity. The strict definition then applies to the scaled copy of Λ of determinant 1; under (2) we may restrict ourselves to isometries by fixing the minima.]

All lattices in dimension 2 (well-rounded or not) are isodual of symplectic type. Thus the classification of isodual classes is non-trivial only if $n \ge 3$.

Proposition 6.3. (1) *If $n = 3$ there does not exist any isodual minimal class.*

(2) *If $n = 4$, the isodual minimal classes are $S4a9$ and $S4a10$, and they are strongly isodual.*

(3) *The only strongly isodual class in dimensions $n = 5, 6, 7$ is the class of the perfect, 7-modular lattice P_6^5.*

Proof. We first consider a perfect class $\mathcal{C}$ (so that there is no difference between weak and strong isoduality). An inspection of perfect lattices shows that the only examples are the classes of $P_4^1 = \mathbb{D}_4$ and and of the 7-modular lattice P_6^5.

Next if $\mathcal{C}$ is a strictly isodual class, then any class $\mathcal{C}' \prec \mathcal{C}$ also is (σ extends to $\overline{\mathcal{C}}$, which contains $\mathcal{C}'$). In particular the vertices of the domain of an isodual class must be isodual lattices. This proves (3) for $n = 5$ and 7.

Let now $\mathcal{C}$ be a non-perfect, isodual class. Then the edges of its domain are Voronoi paths connecting isodual lattices. Since P_6^5 is only connected with $P_6^1 = \mathbb{E}_6$, this is impossible if $n = 6$, which completes the proof of (3), and we are left with the Voronoi path $\mathbb{D}_4$—$\mathbb{D}_4$.

Finally, using explicit computations (see Section 7 below), we check that if $n = 3$ and $s(\mathcal{C}) \le 5$, or if $n = 4$ and either $\mathcal{C} = S4b9$ or $s(\mathcal{C}) \le 8$, then there exists in $\mathcal{C}$ a lattice Λ with $s(\Lambda^*) = 1$ or 2, so that $s(\Lambda^*)$ is

strictly smaller than $s(\mathcal{C}) \geq n$ (because $\mathcal{C}$ is well-rounded).
[Indeed this result holds without assuming that $\mathcal{C}$ is well-rounded.] $\square$

Examples of weakly, non-strongly isodual classes are provided by the Voronoi paths $\mathbb{E}_6$—$\mathbb{E}_6^*$ and $\mathbb{E}_7$—$\mathbb{E}_7^*$ (the extensions of the isoduality to the closure exchanges end-points). Also isodual dual-minimal classes exist in dimensions 3 and 4.

6.2. **One-Part Classes.** We now consider cases in which a minimal class $\mathcal{C}$ is itself a dual-minimal class.

(a) This is trivial if $\mathcal{C}$ is perfect (these are of dimension 0). If $n \leq 4$, this applies to $S2a3$, $S3a6$, $S4a10$ and $S4b10$.

(b) This is also clear if $\mathcal{C}$ is isodual. If $n \leq 4$, this applies again to $S2a3$ and $S4a10$, and also to $S2a2$ and $S4a9$.

(c) If a lattice Λ in $\mathcal{C}$ contains a critical hyperplane section having the same minimum, then the minimal vectors of Λ^* are the primitive vectors orthogonal to this section, and $\mathcal{C}$ does not split whenever $\mathrm{Aut}(\mathcal{C})$ permutes transitively these sections. This is true if $n \leq 4$, and then applies to $S2a$ (again), $S3a5$ and $S4b9$, with $s^* = 2$, and to $S3b4$, $S4b8$ and $S4d7$, with $s^* = 1$.

We are left with 2 classes for $n = 3$ and 12 for $n = 4$. It will turn out that these remaining 14 minimal classes split into at least two dual-classes (and even at least three except for $S4a8$).

6.3. **The method.** Given a minimal class $\mathcal{C}$, we must first find a set T^* of vectors of $\mathbb{Z}^n$, as small as possible, such that for any $\Lambda \in \mathcal{C}$, the components on a basis for Λ^* on the vectors of the dual basis of the chosen basis for Λ are contained in T^*. The theorem below ([M], Lemma 6.3.3 and the comments which follow; see also [Mcpl], Section 14.5.C for the bound $\gamma_9' < 2$) allows such a restriction on T^* :

Theorem 6.4. *Let Λ be a lattice of dimension $n \leq 9$, and let $x \in S(\Lambda)$ and $y \in S(\Lambda^*)$. Then $|x \cdot y| = 0$ or ± 1, except if $n = 8$, Λ is similar to $\mathbb{E}_8$, and x, y are colinear.*

Proof. (*Sketch.*) First observe that $x \cdot y$ is an integer, so that it suffices to prove that $|x \cdot y| < 2$. Consider the chain of inequalities

$$|x \cdot y| \leq \gamma'(\Lambda) \leq \gamma_n' \leq g_n \,.$$

For $n \leq 7$ we have $\gamma_n < 2$, hence the result in this case. For $n = 8$, we have $\gamma_8' = \gamma_8 = 2$, but we may use the strict inequality $|x \cdot y| \leq \gamma'(\Lambda)$ whenever x, y are not colinear. Finally, for $n = 9$, we use the bound $\gamma_9' \leq 2$, relying on the determination by Poor and Yuan of γ_5' and of the 5-dimensional dual-critical lattices combined with inequalities of

Bergé-Martinet ([M], Section 2.8), and on the inspection of possible cases of equality. □

When lattices in $\mathcal{C}$ have a basis $\mathcal{B} = (e_1, \dots, e_n)$ of minimal vectors, since $e_i \cdot \sum_j a_j e_j^* = a_i$, Theorem 6.4 shows that $a_i = 0, \pm 1$. It seems that (at least for low dimensions) more restrictions exist. For instance, for the class $S3a3$, Theorem 6.4 allows $\frac{3^n-1}{2} = 13$ pairs of minimal vectors, whereas only the $\pm e_i^*$ do occur (see Subsection 6.4): here, only vectors orthogonal to a hyperplane section generated by minimal vectors in $\mathcal{C}$ are allowed. This is not general (for instance, in $S4a4$, there exists lattices with $s^* = 5, 6$ or 8); however I have observed that the existence of a rather large number of minimal vectors in $\mathcal{C}$ seems necessary, especially if we limit ourselves to lattices with $s^* \le s$, though I am not able to give this remark a precise formulation.

By lack of better theoretical results, we must try to limit the number of candidates for S^* inside T^* by comparing the norms of various vectors of T^*. Not that inequalities $N(x) \le N(y)$ for $x, y \in T^*$ involve multivariate polynomial (k variables x_i, k denoting the perfection co-rank), for which no general method seems available. Even testing equalities $N(x) = N(y)$ involves the consideration of points (algebraic or rational) on algebraic varieties. In the study of dual-classes with large values of s^*, we met several times conic curves. We cannot exclude to have to consider high genus curves. Note that it may happen that these algebraic varieties are reducible, so that we can reduce ourselves to calculations in lower degrees. This is not general, but occurs for $n = 3$ and $n = 4$, $s \ge 7$, which makes easier the dual classification in these cases.

Recall that $G = \mathrm{Aut}(\mathcal{C})$. On S^* we must consider the action of the transpose group $\widetilde{G}$. This action preserves T^*, that we may divide into orbits under $\widetilde{G}$. To compare the norms of various vectors $x, y \in T^*$, it suffices to choose one vector x in each orbit.

6.4. **Dimension 3.** The classification of dual-minimal classes was done by A.-M. Bergé in [Ber1]; results for well-rounded classes can be read in [M], Section 9.2. We are left with the two classes $S3a4$ and $S3a3$. As an illustration of the techniques used, we give some details.

The minimal-class $S3a4$**.** The group G may be identified with $\{\pm \mathrm{Id}\} \times \mathfrak{S}_4$, $\mathfrak{S}_4$ permuting e_1, e_2, e_3, and $e_4 := -(e_1, e_2, e_3)$. The transposition (e_3, e_4) induces on $\widetilde{G}$ the map $(e_1^*, e_2^*, e_3^*) \mapsto (e_1^* - e_2^*, e_2^* - e_3^*, -e_3^*)$. As a consequence, T^* splits into the two orbits

$$o_1 := \pm\{e_1^*, e_2^*, e_3^*, e_1^* - e_2^*, e_2^* - e_3^*, e_3^* - e_1^*\} \text{ and } o_2 = \pm\{e_1^* \pm (e_2^* - e_3^*)\}\,.$$

It turns out that S^* is contained in o_1, and that the subgroup G_0 of isometries of G exchanges in pairs e_i^* and $e_j^* - e_k^*$, so that $s^* = 2, 4$ or 6.

The three equations $N(e_i^*) = N(e_j^*)$, restricted to the domain $\mathcal{D}$ of $S3a4$, define the three intersection $\mathcal{D} \cap D_i$ of $\mathcal{D}$ with the three lines

$$D_1 \,:\, x_1 = x_2\,; \quad D_2 \,:\, 2x_1 + x_2 + 2 = 0\,; \quad D_3 \,:\, x_1 + 2x_2 + 2 = 0\,.$$

These lines intersect at the point $P := x_1 = x_2 = -\frac{2}{3}$, which defines a dual-minimal class with $s^* = 6$, the similarity class of $\mathbb{A}_3^*$. Off the lines D_i, we have $s^* = 2$, with S^* a permutation of $\{e_1^*, e_2^* - e_3^*\}$. On any $D_i \smallsetminus \{P\}$, we have $s^* = 4$ and S^* is a permutation of $\{e_3^*, e_1^* - e_2^*, e_2^*, e_1^* - e_3^*\}$, obtained on D_1. This splits $\mathcal{D}$ into three dual-minimal classes indexed by s^* and denoted by $S3a4^{(k)}$, $k = 2, 4, 6$.

Here are three remarks on this class.

(1) $S3a4^{(6)}$ disconnects $S3a4^{(4)}$, and similarly $S3a4^{(4)} \cup S3a4^{(6)}$ disconnects $S3a4^{(2)}$.
(2) The map $t \mapsto \frac{2t+4}{t-2}$ is an involution which maps $M2a4(t,t)$ onto a matrix proportional to its inverse. Hence $S3a4^{(2)}$ is weakly isodual.
(3) The fix point of this involution in $\mathcal{D}$ is $t = 2((1-\sqrt{2})$. This value defines the unique isodual lattice in $S3a4$, the lattice named *ccc* (or *mcc*), discovered in [B-M1] and proved in [C-S9] to be the densest isodual lattice in dimension 3, a result that we shall recover below.
(4) The *ccc* lattice is dual-eutactic, but the projections onto $S(ccc)$ and $S(ccc^*)$ span the same the same 4-dimensional subspace of $\mathrm{End}^s(E)$, so that *ccc is not* dual-perfect.
(5) The unique eutactic lattice in $S3a4$ is A_3^*, but this class contains *two* dual-eutactic lattices, namely *ccc* and A_3^*.

The minimal-class $S3a3$**.** This is the class of $\mathbb{Z}^3$, hence $G = 2^3 \cdot \mathfrak{S}_3$. There are three orbits of G on T^*, namely

$$o_1 = \{\pm e_i^*\}\,, \quad o_2 = \{\pm e_i^* \pm e_j^*\}\,, \ \text{ and } \ o_3 = \{\pm e_1^* \pm e_2^* \pm e_3^*\}\,,$$

but as often, only the o_1 (the orthogonal of which has 2 minimal vectors) can be $S(\Lambda^*)$ for some $\Lambda \in S3a3$. The dual-minimal classes are classified by the value of $s^* \in \{1, 2, 3\}$. We denote them by $S3a3^{(i)}$, $i = 1, 2, 3$.

Up to equivalence, representatives for the dual-class $S3a3^{(3)}$ are obtained by setting $x_2 = x_3 = x_1$. The involution $x_1 \mapsto \frac{-2 * x_1}{x_1 + 2}$ shows that $S3a3^{(3)}$ is weakly isodual (we must have $-2/3 < x_1 < 1$). The only fixed point is $x_1 = 0$, defining $\mathbb{Z}^3$, the only isodual lattice of $S3a3$.

As a consequence we see that there are exactly four dual-eutactic lattices in dimension 3, namely A_3, A_3^*, *ccc* and $\mathbb{Z}^3$. We recover the fact that the first two are the only dual-extreme lattices in dimension 3, since this property requires $s + s^* \geq \frac{n(n+1)}{2} = 7$ and *ccc* is not dual-perfect.

6.5. **Dimension 4.** We briefly report on the known results in dimension 4. Discarding the minimal-classes which are at the same time dual-minimal classes, we are left with $S4a8$, $S4a7$, $S4ab$, $S4ac$, and the $4 + 3 + 1 = 8$ classes with $= 6, 5$ or 4. The published results concern only classes with $s = 8$, thus only $S4a8$ in the list above ([M], Exercise 9.5.1).

The class $S4a8$ splits into two dual-minimal classes, $S4a8^{(2)}$ and $S4a8^{(4)}$. This last dual-minimal class contains a unique dual-eutactic lattice ("L_8"), with field of definition the cubic field with discriminant -244; it also contains the eutactic lattice of $S4a8$ (not dual-eutactic), with field of definition $\mathbb{Q}(\sqrt{3})$.

The only proved new result is that lattices with $s = 7$ all have $s^* \leq 3$. The details will be given in the file *Partition.txt* in construction.

As a consequence, there exist exactly four dual-eutactic lattices with $s \geq 7$, namely $\mathbb{D}_4$ and $\mathbb{A}_4$, which are dual-extreme, and $\mathbb{A}_2 \otimes \mathbb{A}_2$ and L_8, which are not. Other known dual-eutactic lattices are $\mathbb{A}_2 \perp \mathbb{A}_2 \in S4a_6$, $A_4^* \in S4a5$, and $\mathbb{Z}^3 \in S4a4$, among which only $\mathbb{A}_4^*$ is dual-extreme.

The proof that $\mathbb{D}_4$, $\mathbb{A}_4$ and $\mathbb{A}_4^*$ are the only dual-extreme lattices ([B-M1]) relies on calculations of extrema. If we could prove that $\mathbb{A}_2 \perp \mathbb{A}_2$ is the only dual-eutactic lattice with $s = 6$, we would have an alternative proof of the classification of dual-extreme lattices.

7. Dual-Minimal Classes: Experimental Results

We first describe general experimental procedure, then concentrate on dimension 4.

7.1. **The method.** We only consider non-perfect classes. Matrices $M_{\mathcal{C}}$ which describe a given minimal-class $\mathcal{C}$ then depend on $k \geq 1$ parameters $x_1, \dots, x_k$, such that $(x_1, \dots, x_k)$ belongs to the chosen domain $\mathcal{D}$ of $\mathcal{C}$.

Choose an integer $m \geq 1$ and consider the matrix M obtained by replacing $x_1, \dots, x_k$ by $\frac{i_1}{m}, \dots, \frac{i_k}{m}$ in $M_{\mathcal{C}}$. Letting (i_ℓ) run through system of integers such that $\frac{i_\ell}{m}) \in \mathcal{D}$, we obtain matrices which define lattices in $\mathcal{C}$. In practice, we rescale M to the integral matrix mM, and since we only need to consider matrices up to equivalence, we make use of reduced domains as constructed in *domains.txt*; moreover, we may restrict ourselves to systems with $\gcd(i_\ell, m) = 1$ and $\gcd(i_\ell) = 1$ or 2, and divide out mM by 2 in the latter case, so as to work with primitive, integral matrices.

We now consider classes of dimension 4 with $s = 6, 5, 4$. In a first step, we calculate s^* for various values of m and put them in each case in a vector of length 12 ($n = 4$ implies $s \le 12$). Note that:

(1) $s^* = 11$ does no exist.
(2) Since we have excluded perfect classes, $s^* = 12$ never occurs, and $s^* = 10$ occurs only for integral lattices in $S4a5$ of minimum 3.
(3) The known results for $s = 9, 8, 7$ show that $s^* = 9$ and $s^* = 7$ will never occur, and that $s^* = 8$ is possible only on $S4a4$.

[Similarly, investigations of cases where $s^* = 6$, then $s^* = 5$, will throw light on large values of s^* on classes with $s = 5$ or 4.]

Inspection of the list of values of s^* for large values of m shows the occurrence of a huge number of examples with $s^* = 1, 2, 3$ (except for restriction coming from non-trivial isometries or relations between norms)), especially for $s^* = 1$, and a very much smaller number of examples with $s^* \ge 4$, if any. Some values of s^* were found only with sparse, large values of m, a phenomenon which could have been forecast for $s^* = 8$ on $S4a4$, thanks to our knowledge of the class $S4a8$; and I cannot exclude that some dual-minimal classes only contain irrational lattices, hence cannot be found experimentally as above.

The first experiments must be extended in order to be able to set precise conjectures. However the picture for $s = 6$ seems to be reliable, with the following possible values for s^*:

$$S4a6\colon\ 1,2,3,4,5\,;\ S4b6\colon\ 1,2\,;\ S4c6\colon\ 1,2,3\,;\ S4d6\colon\ 1,2,3,4,6\,.$$

The next task consists in looking more closely at the sets of minimal vectors which show up, by listing their components on the basis $\mathcal{B}^*$. We first determine T^* and its decomposition into orbits under the action of $\widetilde{G}$, then determine the distribution of vectors in S^* among the different orbits. For instance, if there are two orbits o_1 and o_2 and if $s^* = 1$ the unique pair $\pm x \in S^*$ may belong always, say, to o_1, or to o_1 or o_2, depending on $\Lambda \in \mathcal{C}$, showing in this case the existence of two dual-minimal classes in $\mathcal{C}$; and if $s^* = 2$, we may find one, two or three dual-minimal classes in $\mathcal{C}$.

Let us return to $S4d6$. There are two orbits o_1, o_2, representing $6 + 18 = 24$ pairs of vectors among the 40 pairs of vectors with components $0, \pm 1$, characterized by the number of orthogonal pairs in S

(4 for o_1, 2 for o_2). This suggests that o_1 should play major rôle in the distribution of vectors of S^*, and indeed, we have never found vectors of S^* in o_2. Thus, *experimentally*, we have

$$S^* \subset o_1 := \{\pm e_1^*, \pm e_2^*, \pm e_3^*, \pm e_4^*, \pm(e_1^* + e_2^*), \pm(e_3^* + e_4^*)\}\,.$$

We then easily guess that $S^* = o_1$ holds when we set $x_3 = x_2$ and $x_4 = -x_1 - x_2$ in $M4a6$ (matrix that we shall denote by M_6), and uniquely in this case except for a few matrices corresponding to points on the boundary of $R4a6$ in $D4a6$. We can now *prove* that the two-parameters matrices M_6 for $(x_1, x_2, x_2, -x_1 - x_2) \in D4a6$ do represent a dual-minimal class $S4d6^{(6)} \subset S4d6$, which is thus very likely the only dual-minimal class with $s^* = 6$. Clearly o_1 is a set of minimal vectors for the class $S4d6$. We can prove more:

Proposition 7.1. *The dual-minimal class $S4d6^{(6)}$ is strongly isodual of symplectic type.*

Proof. Consider the matrices M_6 and P:

$$M_6 = \begin{pmatrix} 2 & -1 & x_1 & x_2 \\ -1 & 2 & x_2 & -x_1-x_2 \\ x_1 & x_2 & 2 & -1 \\ x_2 & -x_1-x_2 & -1 & 2 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and check the two equalities

$$P\,M_6\,P = (3 - (x_1^2 + x_2x_1 + x_2^2))\,M_6^{-1} \quad \text{and} \quad P^2 = I_4\,. \quad \square$$

Similarly, experimentation with s^*4 on the reduced domain $Rad6$ output $S^* = \{\pm(e_1^* - e_2^*), \pm e_2^*, \pm(e_3^* - e_4^*), \pm e_4^*\}$, obtained with $x_3 = x_2$, $x4 = -2x_2$. But since S^* is no longer maximal, the result depends on the choice of the reduced domain. A more convenient choice could have been

$$S^* = \{\pm e_1^*, \pm e_2^*, \pm e_3^*, \pm e_4^*\}, \text{ with } x_3 = x_2 \text{ and } x_4 = x_1\}\,.$$

We can prove that these conditions imply that S^* is equal to the set above or to the whole orbit o_1. With the restriction $s^* \le 4$ we define a dual-minimal class $S4d6^{(4)}$. This amounts to exclude the line $x_4 = x_3 = x_2 = -2x_1$.

In a search for dual-eutactic lattices under the conditions $x_3 = x_2$ and either $x_4 = -x_1 - x_2$ or $x4 = -2x_2$ we found only $A_2 \perp \mathbb{A}_2$ (reducible, hence not dual-extreme). This shows that this lattice is the only dual-eutactic lattice in $S4d6^{(6)}$, and that $S4d6^{(4)}$ does not contain any dual-eutactic lattice. The existence of other dual-eutactic lattices in $S4d6$ is very unlikely.

Provisional end of file (except for references)

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