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WEB PAGES ON THE VORONOI ALGORITHM

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Abstract. The first 2 *PARI-GP* files below give the sets of vertices and edges of the Voronoi graph first in dimensions 2 to 6, then in dimension 7. The numerical data are extracted from Jaquet's thesis [Ja]. The third file, based on Chapters 9 and 14 of [M], is devoted to minimal classes in dimensions 2 to 4. We present below a short account of Voronoi's theory and minimal classes.

1. The perfection rank

Let $S \subset \mathbb{R}^n$ $(n \geq 2)$ be a set of non-zero vectors viewed as column-matrices, invariant under the symmetry $x \mapsto -x$. Unless otherwise stated, we assume that S is finite, and we set $s = \frac{1}{2}|S|$. With S we associate the set of the s matrices $M = X{}^tX$ for $X \in S$ (note that X and $-X$ define the same matrix M) and its span V_S in the space $\mathrm{Sym}_n(\mathbb{R})$ of real, $n \times n$ symmetric matrices
[The symmetric matrices $X{}^tX$ may be viewed as *projection matrices*: indeed, let E be an n-dimensional Euclidean vector space equipped with a basis $\mathcal{B}$; if X is the set of components on $\mathcal{B}$ of some vector $x \in E$, then $X{}^tX$ is the matrix in the <u>pair</u> of bases $(\mathcal{B}^*, \mathcal{B})$ of the orthogonal projection $p_x \in \mathrm{End}^s(E)$ to x.]

Definition 1.1. The rank in $\mathrm{Sym}_n(\mathbb{R})$ of the set $\{X{}^tX\}_{X\in S}$ is called the *perfection rank of* S and denoted by $\operatorname{perf} S$ The *perfection rank of a lattice* Λ or a *positive definite quadratic form* Q *on* $\mathbb{R}^n$ (or the symmetric matrix A such that $Q(X) = {}^tXAX$) is the perfection rank of its set of minimal vectors.

We say that Λ or Q (or A) is *perfect* if its perfection rank attains its maximal possible value, namely $\dim \mathrm{Sym}_n(\mathbb{R}) = \frac{n(n+1)}{2}$.

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2. The facets

In what follows, the space $\mathrm{Sym}_n(\mathbb{R})$ is equipped with the *Voronoi scalar product* $\langle M, N\rangle = \mathrm{Tr}(MN)$. We consider a set S as above and we set $r = \mathrm{perf}\, S$.

Definition 2.1. We say that a subset T of S is *admissible* if it is maximal among the subsets T' of S with $\mathrm{perf}\, T' = \mathrm{perf}\, T$.

We denote by W the orthogonal of V_T in V_S:

$$W = \{M \in V_S,\ \forall X \in T,\ \langle M, X^tX\rangle\, (= {}^tXMX)\ = 0\,.$$

We say that $F \in W$ is a *face vector for T in S* the scalar products $\langle F, X^tX\rangle$ are strictly positive for all $X \in S\smallsetminus T$.

The set $\mathcal{C}_S(T)$ of face vectors for T in S is a cone, called the *positive cone of T in S*.

In what follows, we restrict ourselves to the important case where V_T has codimension 1 in V_S. In this case, the cone $\mathcal{C}_S(T)$ is a half-line or is empty.

Definition 2.2. Suppose that $\mathcal{C}_T$ is not empty. An element of $\mathcal{C}_T$ is called a *facet for T in S*.

Note that any T such that $s(T) = s(S) - 1$ (and $\mathrm{perf}\, T < \mathrm{perf}\, S \Leftrightarrow \mathrm{perf}\, T = \mathrm{perf}\, S - 1$) is a facet. However, this special case is far from being general.

3. Contiguity

This time, we start with a positive definite quadratic form Q, with matrix A and minimum m. (If we are dealing with a lattice Λ, we chose a Gram matrix A for Λ and denote by Q the corresponding quadratic form.) We take for S the set $S(Q)$ of minimal vectors of Q and we chose T as above.

Given $F \in V_S$ and $t \in \mathbb{R}$, let R be the quadratic form with matrix F and let $Q_t = Q + t * R$. For $y \in \mathbb{Z}^n$, the function $t \mapsto Q_t(y)$ is increasing, constant or decreasing according to whether $R(y)$ is positive, zero, or negative.

Chose $F \in V_T^{\perp}$. Then we have $Q_t(x) = m$ for all $x \in T$. If $\mathcal{C}_T = \emptyset$, then for every $t \neq 0$, there exists $y \in S$ such that $Q_t(y) < m$. Otherwise, for $F \in \mathcal{C}_T$ and $t > 0$ and small enough, we have $\min Q_t = m$ (and $S(Q_t) = T$).

Let $\theta \in (0, +\infty)$ be the least upper bound of t for which $\min Q_t = m$. If θ is finite, we can consider the form Q_θ.

Definition 3.1. The form Q_θ above is the *contiguous (or neighbour) form of* Q *relative to* T (or *along* F).

Then $S(Q_\theta)$ contains strictly T. Since T is maximal among the subsets of S of perfection rank $r-1$, we have $\operatorname{perf} S(Q_\theta) \geq r$. The form R is proportional to the difference $Q - Q_\theta$. Denoting by B the matrix of Q_θ and rescaling the parameter t, we may put the the matrices of Q_t in the interval $0 \leq t \leq \theta$ in the form

$$A_{t'} = B + t'\,(B - A)\,,\ 0 \leq t' \leq 1$$

to be used in the *PARI-GP* files.

So far the theory has been established (by Voronoi) only for perfect forms. In this case, the contiguous form exists for all facets and is again perfect. Also, and this is the fundamental result of Voronoi, the contiguity graph is connected. The proofs can be read in [M], Chapter 7, Sections 1, 2, 3.

4. Voronoi Graphs: the Results

The references are those of the bibliography of [M], which can be found on this WEB page.

Perfect lattices were classified up to dimension 5 by Korkine and Zolotareff in [K-Z3] (1877). Their results were recovered by Voronoi ([Vo1], 1907), using the construction of the contiguity graph.

Dimension 6 was dealt with 50 years later by Barnes ([Bar4], 1957), who determined directly the contiguity graph. A direct classification of perfect lattices was produced later by Baranovskii and Ryshkov ([Br-R], 1985). However, they did not published the details of their proof.

Finally the results for dimension 7 were first published by Stacey ([Sta1], 1975), but her proof was not considered as definitive, in particular because the lattices she found were not tested for isometry. The first recognized proof was produced by Jaquet in his Neuchâtel thesis ([Ja2], 1991; see also [Ja5] for a published proof). The reference [Ja2] contains the complete data for dimensions 2 to 7.

In their Bordeaux PhD theses, M. Laïhem (1992), J.-L. Baril and H. Napias (19967) obtained a list of 10770 perfect lattices, which was completed a few months later by C. Batut to a list of 10916 perfect lattices; see [M] and the file “Perfection: An introductory paper about perfect lattices.” in this WEB page. To obtain the complete classification of 8-dimensional perfect lattices looked out of scope of any existing method. Nevertheless, M. Dutour Sikirić, A. Schürmann and F. Vallentin ([D-S-V]) were able to prove in 2005 that the list above is complete, by constructing the Voronoi graph. This Voronoi graph can be obtained on request from the authors.

The 48 perfect lattices in dimensions n≤7.

Dimension	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Perfect	1	1	1	2	3	7	33
Edges		1	1	2	4	18	357

5. Voronoi Graphs: the Files

In the two *PARI-GP* files, devoted to dimensions 2–6 and 7 respectively, Gram matrices for the 47 perfect lattices in dimensions n from 2 to 7 are given as vectors $p\mathrm{n}[i]$.

For $n = 2, 3, 4, 5, 6$, the contiguous forms are displayed as a vector $V\mathrm{n}[j]$ with one index j for each edge of the contiguity graph with endpoint some $p\mathrm{n}[k]$ with $k \geq i$.

For $n = 7$, the contiguous forms are displayed as a vector $V7[i][j]$ for $n = 7$, where the index i is that of a perfect form whose neighbour is *equivalent* to some $p7[k]$ with $k \geq i$, and there is one index j for each edge starting form $p7[i]$. **However**, the components of the vectors $V7[i]$ are such that $j \geq j_0$, the least index for which the contiguous form is a $p7[k]$ for some $k \geq i$.

Finally, for $n \leq 6$, the path $A + t * (B - A)$ defined by $V\mathrm{n}[j]$ is directly available using the command *VR*n$[j]$.

As usual, the files contain other informations which can be read under an editor, for instance *emacs*.

End of File *except* for the reference list

Note. The former file from January 2010 cotained three more sections:
6. *Minimal Classes and (Weak) Eutaxy.*
7. *Minimal Classes: the Results.*
8. *Minimal Classes: the Files.*

The content of theses three sections has been extended in 2017, and is now part of the file *domains.txt* below.

The reference list below contains only five items. For the other papers cited in the text, see the updated reference list of [Mar] on this home page.

References

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[D-S-V] M. Dutour Sikirić, A. Schürmann, F. Vallentin, *Classification of eight dimensional perfect forms*, Electron. Res. Announc. Amer. Math. Soc. **13** (2007), 21–32.

[E-G-S] P. Elbaz-Vincent, H. Gangl, C. Soulé, *Perfect Lattices, Homology of Modular Groups and Algebraic K-theory*, Oberwolfach Reports 1/2005 (2005), 36–39.

[Ja] D-O. Jaquet, *Énumération complète des classes de formes parfaites en dimension* 7, Thèse, Neuchâtel (1991).
[This is reference [Ja2] of Section 4.]

[M] J. Martinet, *Perfect Lattices in Euclidean Spaces*, Grundlehren **327**, Springer-Verlag, Heidelberg (2003).

Other items are given in Section 4 above.