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A2X – WEB Pages on Lattices.

LATTICES and SPHERICAL DESIGNS

Content :

A2X – WEB PAGES on LATTICES and SPHERICAL DESIGNS

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1. General Results on Spherical Designs

The basic reference for the connections between the theory of lattices and that of spherical designs is Venkov's 2001 paper [V].

Definition 1.1. *Let Σ be the unit sphere in a Euclidean space E of dimension $n \geq 1$, endowed with the natural measure d_σ in the scale in which Σ has volume 1. (Hence E could be identified with $\mathbb{R}^n$, as in* [V]*.) Let X be a non-empty, finite subset of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and let t be a positive integer. We say that X is a* spherical t-design *if*

$$\int_{S^{n-1}} f\,dx = \frac{1}{|X|} \sum_{x \in X} f(x)\,.$$

for every polynomial f of degree at most t.

The definition immediately extends to a sphere having an arbitrary radius. Since we shall never consider *combinatorial designs*, we shall often omit the word "spherical".

It is proved in [V] that it suffices to consider the condition above for *harmonic polynomials*, which then reads:

Theorem 1.2. *The set X is a t-design if and only if one has*

$$\sum_{x \in X} f(x) = 0$$

for every non-constant, homogeneous harmonic polynomial f of degree at most t.

Denote by e (resp. $\imath$) the largest even (resp. odd) integer $t' \leq t$. Then:

Theorem 1.3. *The set X is a t-design if and only if there exists a constant c such that*

$$\sum_{x \in X} (x \cdot \alpha)^e = c\,(\alpha \cdot \alpha)^{e/2} (x \cdot x)^{e/2} \quad \text{and} \quad \sum_{x \in X} (x \cdot \alpha)^\imath = 0\,.$$

The value of c then solely depends on n and t, and in practice, one applies the theorem above to all even integers $e \leq t$, using the value below c_e of c:

$$c_e = \frac{1 \cdot 3 \cdot 5 \cdots (p-1)}{n(n+2) \cdots (n+p-2)}\,|X|\,.$$

The following properties are easy consequences of the results above:

- A disjoint union of t-designs is a t-design.[1]
- A t-design is a t'-design for every $t' \leq t$.
- X is a 1-design if and only if it is symmetric (w. r. to the origin).
- Any symmetric $(2t)$-design is a $(2t+1)$-design.
- In dimension $n = 1$, the unique 1-design is S^0 and it is a t-design for all t. Otherwise, there is a maximal value t_{max} for t, the *level of* X, such that a given X is not a t-design for any $t > t_{\max}$.

The theory of invariants allows the construction of many interesting designs. Let G be a finite subgroup of the orthogonal group $\mathrm{O}(E)$. The polynomial functions (on E) which are invariant under G constitute an algebra over $\mathbb{R}$ which is known to be finitely generated. (The degrees of conveniently chosen generators are called the *fundamental degrees.*) The polynomials $(x \cdot x)^k$ are examples of degree $2k$ which are invariants under all groups $G \subset \mathrm{O}(E)$. Given a finite set X invariant under a group G, the polynomial $\sum_{v \in X} (v \cdot x)^k$, if non-zero, is an invariant polynomial of degree k.

The following theorem is an immediate consequence of theorem 1.3:

Theorem 1.4. *Let $t = 2p$ be an even integer and let G be a subgroup of $\mathrm{O}(E)$ whose only invariants of degree $d \leq t$ are polynomials in $x \cdot x$. Then the orbit under G of any non-zero vector of E is a spherical t-design, and thus more generally, any finite, non-empty invariant subset of a sphere is a t-design.*

The case of 2-designs deserves some comments: the condition of the theorem above is equivalent to the fact that the representation of G afforded by E is irreducible over $\mathbb{R}$ ("Brauer-Coxeter condition"; see Section 2).

Moreover, the notion of a 2-design may be viewed as a special case of the *eutaxy property.* Let us say that a finite, symmetric subset X of a sphere is *weakly eutactic* if there exists an identity

$$\sum_{x \in X} \rho_x \, (x \cdot \alpha)^2 = (\alpha \cdot \alpha) \, (x \cdot x)$$

with real coefficients ρ_x, that it is *eutactic* if the can choose strictly positive ρ_x, and *strongly eutactic* if there exists such an identity with equal ρ_x. Then X is a 2-design if and only if it is strongly eutactic. In practice, we shall write eutaxy relations as sums $\sum_{x \in X/\{\pm\}}$ over a system of representatives for X modulo central symmetry.

[1] This may be a t'-design for some $t' > t$. Examples provided by the set of minimal vectors of a lattice and a scaled copy of its dual are proved in [Bc-V]

2. Designs and Lattices

From now on, we apply the results of Section 1, taking for X the set $S = S(\Lambda)$ of minimal vectors of a lattice $\Lambda \subset E$, or more generally other layers of Λ.

For $x \in E \smallsetminus \{0\}$, denote by $p_x \in End^s(E)$ the orthogonal projection to the line $\mathbb{R}x$. Recall (see [M], Chapter 3) that a lattice is *perfect* if the p_x, $x \in S(\Lambda)$ span $\mathrm{End}^s(E)$, that a *eutaxy relation for* Λ is an equality

$$\mathrm{Id} = \sum_{x \in S(\Lambda)} \rho_x \, p_x \,,$$

that Λ is *weakly eutactic* if it possesses a eutaxy relation, *eutactic* if it possesses such a relation with strictly positive coefficients ρ_x, and that Λ is *extreme* if the Hermite invariant attains a local maximal at Λ. A well known theorem of Voronoi (see [M], Theorem 3.4.6) reads

$$\Lambda \text{ extreme} \iff \Lambda \text{ perfect } \& \ \Lambda \text{ eutactic}\,.$$

Definition 2.1. *We say that* Λ *is* strongly eutactic *if* $S(\Lambda)$ *is a* 2*-design (or* 3*-design, this amounts to the same) and that* Λ *is* strongly perfect *if* $S(\Lambda)$ *is a* 4*-design (or* 5*-design, this amounts to the same).*

It is clear that Λ is strongly eutactic if and only if $S(\Lambda)$ is strongly eutactic in the sense of Section 1. The following theorem can be used to prove that certain lattices are perfect, whereas direct proofs are sometimes not available.

Theorem 2.2. (Venkov) *A strongly perfect lattice is extreme and strongly eutactic.*

Proof. Two proofs, due to Boris Venkov and Thierry Vust, are given in [V], Section 6. □

The fact that the set of minimal vectors of a given lattice Λ is or is not a t-design for some given t can be tested using directly the definition, in the form given in [V], Theorem 8.1. Results for $t = 5$ have been checked for all known strongly perfect lattices of dimension $n \leq 26$ mentioned in [V], Section 19. The calculations become lengthy for large dimensions (and would be much more time consuming if we were to consider larger values of t). As an example, to prove that the integral 26-dimensional lattice of minimum 4 and determinant 3 (found by Gabriele Nebe in her classification of maximal finite subgroups of $\mathrm{GL}_{26}(\mathbb{Z})$) took 41 hours, 41 minutes, 41 seconds[2] (and some milliseconds) on a *Dell* station.

[2] Happily, we do not believe in numerology!

Applying the theorem below, Christine Bachoc obtained the result in a more general form and a much shorter time.

Theorem 2.3. *Let $t = 2p$ be an even integer and let Λ be a lattice such that the only invariants of degree $d \leq t$ of its automorphism group are polynomials in $x \cdot x$. Then all layers of Λ and of its dual lattice Λ^* are t-designs.*

Proof. This is a direct consequence of Theorem 1.4, taking into account that $\mathrm{Aut}(\Lambda) = \mathrm{Aut}(\Lambda)^*$. □

Another important source to prove that some lattices are strongly perfect is to use the theory of modular forms. This theory is of a particular interest in the study of even modular lattices. Recall that a lattice Λ is *ℓ-modular* if there exists an isometry of Λ onto $\sqrt{\ell}\,\Lambda^*$ and that it is *modular* if it is ℓ-modular for some ℓ. (The modular lattices are the integral isodual lattices.) The *(narrow) level* of an even lattice Λ is the smallest integer ℓ such that $\sqrt{\ell}\,\Lambda^*$ is even. Even modular lattices of narrow level a prime ℓ such that $\ell + 1$ divides 24 (or $\ell = 1$; these are merely the even unimodular lattices) have been considered by Quebbemann, who proved in [Q] that the theta series of such a lattice is modular for the *Fricke group*, a group which contains $\Gamma_0(\ell)$, and using this device, that we then have $\min \Lambda \leq 2 + 2\left\lfloor \frac{(\ell+1)\,n}{48} \right\rfloor$. Lattices which meet this bound are called *extremal*. In [Bc-V], Bachoc and Venkov obtained theorems of strong eutaxy or perfection for $\ell = 1, 2, 3, 5$. Here is a summary of their results. [3]

Theorem 2.4. (Bachoc–Venkov) *The layers of an extremal ℓ-modular lattice are t-designs according to the data below:*

(1) $\ell = 1$ & $n \equiv 0 \mod 24 \Longrightarrow t \geq 11$.
(2) $\ell = 1$ & $n \equiv 8 \mod 24 \Longrightarrow t \geq 7$;
$\ell = 2$ & $n \equiv 0 \mod 16 \Longrightarrow t \geq 7$.
(3) $\ell = 2$ & $n \equiv 4 \mod 16 \Longrightarrow t \geq 5$;
$\ell = 3$ & $n \equiv 0, 2 \mod 12 \Longrightarrow t \geq 5$;
$\ell = 5$ & $n = 16 \Longrightarrow t = 5$.
(4) $\ell = 1$ & $n \equiv 16 \mod 24 \Longrightarrow t \geq 3$;
$\ell = 2$ & $n \equiv 8 \mod 16 \Longrightarrow t \geq 3$;
$\ell = 3$ & $n \equiv 4, 6 \mod 12 \Longrightarrow t \geq 3$.

[In all known examples, the lower bounds displayed above are sharp. We conjecture this is a general fact.]

[3] The assertion for $\ell = 1$ is an older result of Boris Venkov

3. Strongly Eutactic Lattices

As always in lattice theory, two problems arise:

(1) To construct strongly eutactic lattices.
(2) To classify strongly eutactic lattices.

In this section, we also consider a slightly less restrictive property. Recall that a lattice is called *semi-eutactic* if it possesses a eutaxy relation with non-negative coefficients.

Definition 3.1. *We say that a lattice is* strongly semi-eutactic *if it possesses a eutaxy relation with equal non-zero coefficients.*

Such a relation is of the form $\mathrm{Id} = c\,\sum_{x\in S'/\{\pm\}} p_x$ where $S' \subset S(\Lambda)$ is the set of minimal vectors of Λ whose eutaxy coefficients are non-zero (and $c = \frac{n}{s'}$ with $s' = \frac{1}{2}\,|S'|$). Then S' is a spherical 2-design.

To classify strongly eutactic lattices using Theorem 1.3 looks difficult, even in, say, dimension 4. Classification results in low-dimensions have been obtained using a different concept.

We say that a lattice is *well rounded* (W.R. for short) if its minimal vectors span E. It immediately follows from the definition that a weakly eutactic lattice is well rounded. We consider on the space $\mathcal{E}$ of W.R. lattices the relation

$$\Lambda \sim \Lambda' \Longleftrightarrow \exists u \in \mathrm{GL}(E),\ \Lambda' = u(\Lambda) \ \text{ and }\ S(\Lambda') = u(S(\Lambda))$$

(see [M], Chapter 9). We obtain this way a partition of $\mathcal{E}$ into finitely many *minimal classes*, related to an (infinite) cellular decomposition of the space of positive definite quadratic forms having a given minimum: minimal classes correspond to cells up to equivalence under $\mathrm{GL}_n(\mathbb{Z})$, and cells are convex polyhedrons in $\mathbb{R}^{n(n+1)/2}$. By a theorem of Bergé–Martinet, each cell contains at most one weakly eutactic form. This shows that in a given dimension n, the set of weakly eutactic lattices up to similarity is finite, and in particular that there are up to similarity only finitely many strongly eutactic lattices.

Classification up to dimension 4 (Štogrin, Bergé–Martinet) together with the list of weakly eutactic lattices can be read in [M], Sections 9.3 and 14.3. Dimension 5 was completely dealt with by Batut in [Bt]. In dimension 6, the minimal classes (but not the eutactic lattices) have been classified by Elbaz-Vincent, Gangl and Soulé, who recently also solved the case of dimension 7 ([E-G-S]; see also [E]).

Using this device one could complete the classification of 6-dimensional strongly eutactic lattices. Indeed, given a class $\mathcal{C}$ and a lattice $\Lambda \in \mathcal{C}$, let $T \in \mathrm{Mat}_{n,s}(\mathbb{Z})$ be the matrix of components of $S(\Lambda)$ on some basis $\mathcal{B}$ for Λ, and define the *Bacher matrix of* $\mathcal{B}$ *and* Λ (nowadays more usually named the *barycentre matrix*) by $\mathrm{Bc} = T * {}^tT \in \mathrm{GL}_n(\mathbb{Z})$. The equivalence class of a Bacher matrix for $\mathcal{C}$ (called the *barycentre matrix* in [E]) characterises the class $\mathcal{C}$. Moreover, the definition of Bc shows that

$$\mathrm{Bc}^{-1} \in \mathcal{C} \Longleftrightarrow \mathrm{Bc}^{-1}\ \textit{is a Gram matrix for a strongly eutactic lattice in}\ \mathcal{C}\,.$$

More generally, given a class $\mathcal{C}_0$ with set of minimal vectors S_0 and Bacher matrix $\mathrm{Bc}_0 = S_0\,{}^tS_0$, if $S = S(\mathrm{Bc}_0^{-1}) \supset S_0$ then S is the set of minimal vectors of a class $\mathcal{C}'$ with set of minimal vectors S and Bc^{-1} is a strongly semi-eutactic matrix belonging to $\mathcal{C}'$, for which S_0 is the set of minimal vectors having non-zero eutaxy coefficients. Strongly semi-eutactic, 6-dimensional lattices have been searched in a list of Bacher matrices by simply checking the (rather scarce) inequality $s(\mathrm{Bc}^{-1}) \ge s(\mathcal{C})$. This gives the list of strongly semi-eutactic *for which minimal vectors with non-zero coefficients define a minimal class.* This is certainly true if whenever the perfection rank is equal to the kissing number, but I cannot assert that this property is general for strongly semi-eutactic lattices.

As in dimension 6, the classification of strongly eutactic lattices in dimension 7 *could* be extracted from the numerical data obtained by Elbaz-Vincent, Gangl and Soulé in their paper [E-G-S].

Here are other methods (besides group theory, modular forms and minimal classes) for constructing strongly eutactic lattices:

- Algebraic constructions (orthogonal sums, tensor products, exterior powers).
- Classification for small values of s (Bergé-Martinet; see [Be-M2])
- The relation lattice ([M-V], Section 5).
- Convenient sections of strongly perfect lattices ([M-V], Section 6).
- Also, infinite series can be guessed (and the guess checked in not too high dimensions) by extrapolation from very low dimensions.

For example, consider for $n \ge 5$ odd the $n \times n$ matrix M with entries $n-1$ on the diagonal and -1 off the diagonal, except for entries 0 at $(i, i+1)$ and $(i+1, i)$, $i = 1, 3, \ldots, n-2$. Then, *experimentally*, M^{-1} is the Gram matrix for a strongly eutactic lattice Λ_n. In the scale which makes Λ_n integral and primitive, we should have $\min \Lambda_n = 2(n-1)$, $s = (n^2-1)/2$, $\mathrm{perf} = s$, $\mathrm{Ann}(L_n^*/L_n) = n^2 - 1$, and even $\mathrm{Smith}(\Lambda_n) = (n^2-1)^{(n-1)/2} \cdot 2(n+1) \cdot \frac{n+1}{2}$. This accounts for lattices std12, stf24b and sth40 in the table. Note that M lies in the minimal class of $\mathbb{A}_n^*$; a somewhat similar construction produces the lattices std9, stf16 and sth25.

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4. Strongly Perfect Lattices

The main two problems are the same as in section 3. But besides them, it is useful to quote an important specific result, which concerns the *Bergé-Martinet invariant*

$$\gamma'(\Lambda) = \big(\gamma(\Lambda)\,\gamma(\Lambda^*)\big)^{1/2} = \big(\min \Lambda \, \min \Lambda^*\big)^{1/2}\,,$$

for which Venkov proved the lower bound $\gamma' \geq \frac{n+2}{3}$ ([V], Section 10). Calculating this invariant over perfect lattices of dimensions $n \leq 7$ suffices to classify strongly perfect lattices in these dimensions. (Probably, perfect lattices Λ such that $\gamma'(\Lambda) \geq \frac{n+2}{3}$ are still strongly perfect for a few higher dimensions, but this is not general.)

For constructing strongly perfect lattices, besides methods relying on group theory and modular forms that we mentioned in Section 2, we may quote

- Explicit calculations with classical lattices.
- Equiangular families of lines (see classification below).
- Combinatorial properties of binary codes.

Explicit calculations were notably carried out for many sublattices of the Leech lattice. The lattice K'_{21} deserves a special comment: this is the only known strongly perfect lattice Λ such that Λ^* *is not* strongly perfect. Probably, this is merely an illustration of the fact that we do not know how to construct strongly perfect lattices without the help of group theory or modular forms.

Combinatorial properties of binary codes were used by Venkov to handle Barnes-Wall lattices BW_n, $n = 2^p$ (he proved that $S(\mathrm{BW}_n)$ is a 7-design from dimension 8 onwards). This result was recently improved by Bachoc ([Bc]), who showed using group theory that indeed *all layers* of these lattices are 7-designs. (Her work relies on [Ne-R-Sl].)

Classification results concern the dimension, the minimum of integral scaled copies and lattices with $s = \frac{n(n+1)}{2}$.

Theorem 4.1. *The strongly perfect lattices of dimension $n \leq 12$ (up to scale) are $\mathbb{Z}$, $\mathbb{A}_2$, $\mathbb{D}_4$, $\mathbb{E}_6$, $\mathbb{E}_6^*$, $\mathbb{E}_7$, $\mathbb{E}_7^*$, $\mathbb{E}_8$, K'_{10}, ${K'_{10}}^*$ and the Coxeter-Todd lattice K_{12}.*

Proof. See [V] for $n = 1-9$ and $n = 11$, [Ne-V1] for $n = 10$, and [Ne-V2] for $n = 12$.
[Probably, no strongly perfect lattices exist in dimensions 13 and 15, and Souvignier's 3-modular lattice Q_{14} is the only strongly perfect lattice in dimension 14; see [Ne-No-V] and [Ne-V4].] □

For integral lattices of minimum m, it is proved in [V], Prop. 7.14 that $\dim \Lambda \leq 3(m^2-1)$. Lattices of minimum 2 are root lattices. The condition $\gamma'(\Lambda) \geq \frac{n+2}{3}$ easily shows that the list is $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{D}_4$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$. The problem of minimum 3 is much more difficult. It is proved in [V], Section 7 that there are 5 such lattices, the unimodular lattice O_{23}, scaled copies of $\mathbb{Z}$ and $\mathbb{E}_7^*$, and their orthogonal O_{22} and O_{16} in O_{23} ($\mathrm{O}_{16\,even}$ is BW_{16}).

Finally, strongly perfect lattices with minimal s are $\mathbb{Z}$, $\mathbb{A}_2$ and lattices of dimension $n = (2k+1)^2 - 1$ whose minimal vectors constitute an equiangular family of lines; examples are known only for $n = 7$ ($\mathbb{E}_7^*$) and $n = 23$.

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5. Higher designs

Theorem 1.3, which gives only one equation for 3-designs, gives two equations for 5-designs and three equations for 7-designs. In this respect, high designs are easier to handle. The theorem below proved in [M1][4] is an example where a classification of 7-designs is obtained whereas the corresponding result is not known for general strongly perfect lattices (which are only 5-designs):

Theorem 5.1. *The integral primitive lattices* Λ *of minimum* $m \leq 5$ *whose set* $S(\Lambda)$ *of minimal vectors is a spherical* 7*-design are* $\mathbb{Z}$*, the root lattice* $\mathbb{E}_8$*, the shorter Leech lattice* O_{23}*, the three laminated lattices* Λ_{16} *(the Barnes-Wall lattice* BW_{16}*),* Λ_{23} *and* Λ_{24} *(the Leech lattice, an* 11*-design), and the even unimodular lattices of dimension* 32 *and minimum* 4 *(which have not been classified). In particular, minimum* 5 *is not possible.*

More recently, Nebe and Venkov have (almost) obtained ([Ne-V3]) the classification of lattices whose set $S(\Lambda)$ of minimal vectors is a spherical 7-design for lattices of dimension $n \leq 23$ (with a modicum of doubt in dimension 23).

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[4] the lattice O_{23} has been forgotten in the statement written in [M1]

6. The Tables

We present two tables in *PARI-GP* format:

(1) *strongeut.gp*, in which we list known semi-strongly eutactic lattices in dimensions up to 10 ;
(2) *strongperf.gp*, in which we list known strongly perfect lattices in dimensions up to 26.

Strongly perfect lattices of dimension $n \le 10$ occur in both tables. Lattices are given by a Gram matrix. Since some of them occur several times in this WEB page, often with different Gram matrices, they have been systematically given different names.

These two tables have the same structure as those which are displayed in previous tables on perfect lattices:
name=[...];
followed by
\\ $[det, [s, min], [s^*, min], [a_1^{e_1} \dots a_r^{e_r}]]$
and may be some more data (name, number of automorphisms,...) and the cardinality of the afforded design in case the lattice is semi-eutactic, non-eutactic, written s_{design}; as usual, *PARI* does not read the line above, which is here for the reader's convenience.

The first table deserves some more comments. Each name begins with "st" followed by one of the letters a,b,…,i (a for dim. 2, b for dim. 3, ..., i for dim. 10) and the s invariant. When necessary, a letter a,b,c, or d is added to distinguish lattices. Finally, the name ends with "se" if the lattice is semi-eutactic, non-eutactic.

Example with $n = 6$:
ste11se=[8,-1,1,1,-2,-2;-1,8,-2,-2,1,1;1,-2,8,-4,-1,-1;1,-2,-4,8,-1,-1;-2,1,-1,-1,8,-4;-2,1,-1,-1,-4,8];
\\ $[62208, [11, 8], [4, 2], [12^4.3.1]]$ $|AUT| = 256 = 2^8$;
\\ $s_{design} = 9$ orb=(1+8,2)
Here, "orb=(1+8,2)" means that minimal vectors with non-zero eutaxy coefficients share out among two orbits with 1 and 8 elements, and that the remaining two vectors constitute one more orbit.

In the table below, we display the number of known strongly eutactic and semi-eutactic, non-eutactic lattices in dimensions 1 to 6. This is an update[5] of the data published in [M-V] (which however did not mentioned semi-eutaxy).

[5] in [M-V], Table 8.1, a lattice with $n = 5$, $r = s = 12$ and minimum 10 has been forgotten

Low-dimensional strongly eutactic lattices

dimension	1	2	3	4	5	6
well-rounded cells	1	2	5	18	136	5634
eutactic	1	2	5	16	118	??
strongly eutactic	1	2	3	6	9	21
semi-eutactic	0	0	0	1	5	??
strongly semi-eutactic	0	0	0	1	1	≥ 6

The data for dimension 6 have been communicated to us by Elbaz-Vincent and Gangl at the Oberwolfach meeting [ObW]. Beforehand, only 20 strongly eutactic lattices out of 21 were known, 19 listed in [M-V] and 1 in [Be-M1]; and only 4 out of 6 semi-eutactic lattices.

The data for dimension 7 to 10 were obtained using various tricks, including the recent results outlined in [Be-M2], and an exploration of the minimal classes which lie below $\mathbb{E}_7^*$. Work in progress by Elbaz-Vincent, Gangl and Soulé (who have yet considered 20 out of the 33 perfect, 7-dimensional lattices) will produce important enlargements of the to-day available data.

Known low-dimensional strongly perfect lattices

dimension	1	2	3	4	5	6	7	8	9
number	1	1	0	1	0	2	2	1	0
dimension	10	11	12	13	14	15	16	17	18
number	2	0	1	0	1	0	4	0	2
dimension	19	20	21	22	23	24	25	26	27
number	0	3	1	10	7	2	0	3	0

Up to dimension 24, the numbers displayed in the table above correspond to lattices which are described in [V], Tables 19.1 and 19.2. In dimension 26, they correspond to one 3-modular lattice, the integral lattice (Nebe) of minimum 4 and determinant 3, and its dual. All together, there are 44 lattices: 14 pairs (Λ, Λ^*) of non-isometric lattices Λ, Λ^*; 15 isodual lattices; and the lattice K'_{21}.

The results of [V], Section 17 together with King's tables ([K]) imply that there exist numerous strongly eutactic lattices in dimensions 28, 30, 31, 32.

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References

[Bc] C. Bachoc, *Designs, groups and lattices* (talk at the Jounées Arithmétiques of Graz, 2003), J. Th. Nombres de Bordeaux **17** (2005), 25–44.

[Bc-V] C. Bachoc, B. Venkov, *Modular forms, lattices and spherical designs*, [EM], 87–11.

[Bt] C. Batut, *Classification of quintic eutactic forms*, Math. Comp. **70** (2000), 395–417.

[Be-M1] A.-M. Bergé, J. Martinet, *Symmetric Groups and Lattices*, Monatshefte Math. **140** (2003), 179–195.

[Be-M2] A.-M. Bergé, J. Martinet, *Voronoi graphs, cells, and spherical designs*, in [ObW].

[EM] J. Martinet (ed.), *Réseaux euclidiens, designs sphériques et formes modulaires*, L'Ens. Math., Monographie **37**, Genève (2001).

[E] Philippe Elbaz-Vincent, in [ObW].

[E-G-S] Philippe Elbaz-Vincent, Herbert Gangl, Christophe Soulé, *P*erfect forms and the cohomology of modular groups, Advances in Math. **245** (2013), 587–624; preprint at arXiv:math/1001.0789v1; see also [E].

[K] O. King, *A mass formula for unimodular lattices with no roots*, Math. Comp. **72** (2003), 839-863 and *Table of masses of 32-dimensional even unimodular lattices having any given root system*, 2 megabytes, available at http://www.math.berkeley.edu/~king/.

[L-S-T] W. Lempken, B. Schröder, P.H. Tiep, *Symmetric squares, spherical designs, and lattice minima. With an appendix by* C. Bachoc *and* P.H. Tiep, J. Algebra **240** (2001), 185–208.

[M] J. Martinet, *Perfect Lattices in Euclidean Spaces*, Grundlehren **327**, Springer-Verlag, Heidelberg (2003).

[M1] J. Martinet, *Sur certains designs sphériques liés à des réseaux entiers*, [EM], 135–146.

[M-V] J. Martinet, B. Venkov, *Les réseaux fortement eutactiques*, [EM], 112–134.

[Ne-No-V] , G. Nebe, E. Nossek, B. Venkov, *Low dimensional strongly perfect lattices. II: Dual strongly perfect lattices of dimension 13 and 15*, J. Th. Nombres Bordeaux **25** (2013), 147–161.

[Ne-R-Sl] G. Nebe, E. Rains, N.J.A. Sloane, *Codes and Invariant Theory*, Math. Nachrichten 274-275 (2004), 104-116.

[Ne-Sl] G. Nebe, N.J.A. Sloane, *A Catalogue of Lattices*, http://www.research.att.com/~njas/lattices/index.html.

[Ne-V1] G. Nebe, B. Venkov, *The strongly perfect lattices in dimension* 10, J. Th. Nombres de Bordeaux **12** (2000), 503–518.

[Ne-V2] G. Nebe, B. Venkov, *Low dimensional strongly perfect lattices. I: The 12-dimensional case*, L'Enseignement Mathmatiques **51** (2005), 129–163.

[Ne-V3] G. Nebe, B. Venkov, *On lattices whose minimal vectors form a* 6*-design*, preprint (2006); European J. Combin. **30** (2009), 716–724. (Special issue dedicated to Eiichi Bannai's 60th birthday.)

[Ne-V4] G. Nebe, B. Venkov, *Low dimensional strongly perfect lattices. III: Dual strongly perfect lattices in dimension 14*, Inter. J. Number T. **6** (2010), 387–409.

[ObW] *Tagung Gitter und Anwendungen, January 3.-7, 2005*, Oberwolfach Reports, 1 2005.

[Q] H.-G. Quebbemann, *Modular lattices in Euclidean spaces*, J. Number Theory **54** (1995), 190–202.

[S-T] R. Scharlau, P.H. Tiep, *Symplectic group lattices*, Trans. Amer. Math. Soc. 351 (1999), 2101–2139.

[T] P.H. Tiep, in [ObW].

[V] B. Venkov, *Réseaux et designs sphériques (notes by J. Martinet)*, [EM], 10–86.

[V1] B. Venkov, in [ObW].