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UNIVERSITÉ BORDEAUX 1

WEB Pages on Lattices.

PERFECT LATTICES

Content:

Special pages are devoted to strong eutaxy and strong perfection.

1. General Results on Perfection and Low-Dimensional Lattices.

The notion of an *extreme lattice* was introduced (in terms of quadratic forms) by Korkine and Zolotareff ([K-Z2], 1873): these are the lattices on which the Hermite invariant (see below) attains a local maximal. They proved in [K-Z3] that extreme lattices satisfy a certain property, to be called later *perfection* by Voronoï ([Vor]). The basic definitions concerning the notion of perfection can be read in Chapter 3 of [M]. Their aim was to determine the *critical* or *absolutely extreme lattices* for dimension 5, a program previously carried out for $n = 2$ (Lagrange), $n = 3$ (Gauss) and $n = 4$ ([K-Z1]).

The classification of n-dimensional perfect lattices is known up to dimension $n = 8$.

For $n \leq 5$, the results were obtained by Korkine–Zolotareff in 1877 ([K-Z3]), using combinatorial methods, and were recovered by Voronoï in 1907, using the contiguity (or neighbouring) algorithm he invented, which endows the set of perfect quadratic forms up to scale and equivalence with a structure of a finite, *connected* graph (the *Voronoi graph*; [Vor]). Up to dimension 5, perfect lattices are extreme.

The accepted known classification results in larger dimensions were all obtained by constructing the Voronoi graph. Dimension 6 was obtained by Barnes ([Barn]) in 1957. There are 7 perfect lattices, among which 6 are extreme. The existence of a perfect, non-extreme form was known to Voronoi; see the introduction of [Vor]. An alternative, combinatorial proof, based on a method introduced by Watson in his 1971 paper [Wa1], was found in 1985 by Baranovskii and Ryshkov ([Br-R]), but they did not published the details. (In the spring of 2007, Anne-Marie Bergé and myself tried to work out the details of a proof based on the methods of [Wa1]; our estimation was that we needed about 25 pages proof, and we decided to give up writing.)

The first classification proof for dimension 7 was published in 1975 by Kaye Stacey ([St]). Her proof relied on Watson's improvement [Wa2] for dimension 7 of its methods. The work of Stacey was considered as not completely satisfactory, in particular because the test for equivalence between perfect forms was missing (and also because of some slips, the corrections of which were however guessed by Conway and Sloane in [C-S1]). The results of Stacey in the form they have in [C-S1] were confirmed by Jaquet in his 1971 Neuchâtel thesis [Ja1], in which he constructed the Voronoi graph in dimension 7; see also the published paper [Ja2].

A detailed account of the properties of perfect forms in dimensions $n \le 7$ can be read in Conway and Sloane's [C-S1]. Besides a list of Gram matrices, this paper contains various data relative to perfect forms (or lattices) up to dimension 7, including automorphism groups and roots, and proofs for $n \le 4$.

We adopt the notation of [C-S1] for $1 \le n \le 7$: there are 48 perfect lattices (up to similarity), denoted by P_n^i, where the index i takes values between 1 and the number i_n of perfect n-dimensional lattices; more precisely, the notation P_n^i is used for an integral primitive representative of the similarity class. The norm of the lattices P_7^i are thus well defined integers.

Another source is [M], chapter VI, which contains other data as well as proofs for dimensions $n \le 5$. (The proof given in the English edition, relying on methods of Watson, is much shorter than that of the French edition, which followed closely [K-Z3]).

Statistics on the 48 perfect lattices of dimension $n \le 7$.

Dimension	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Perfect	1	1	1	2	3	7	33
Extreme	1	1	1	2	3	6	30
Norms	1	2	2	2	$2, 4$	$2, 4$	$2, 3, 4, 6, 8, 10$
$\frac{1}{2}$ Kiss. Nb.	1	3	6	$10, 12$	$15, 20$	$21, 22, 27, 30, 36$	8 values $(*)$

$(*)$ 28, 29, 30, 32, 34, 36, 42, 63.

The 4 perfect, non-extreme lattices are P_6^4 and P_7^{26}, which are semi-eutactic, and P_7^{18} and P_7^{29}, which are not.

The Voronoi graphs for dimension 2 to 7 can be downloaded from this homepage.

We now turn to dimension 8. Work done in Bordeaux by M. Laïhem ([Lah]), C. Baril ([Bari]), H. Napias ([Nap]), completed by C. Batut (unpublished) produced in 1996 a list of **10916** perfect lattices. Here are some details:

• Laïhem classified those lattices having a perfect 7-dimensional cross-section with the same minimum; he found 1175 such lattices. (He however left aside the case of a section $\mathbb{A}_7$, $\mathbb{D}_7$, or $\mathbb{E}_7$; his proof was later completed by Baril.)

• Baril classified those lattices having a decomposition $P_6 \oplus \mathbb{A}_2$ for some perfect 6-dimensional lattice (scaled to minimum 2), obtaining 53 non-Laïhem lattices.
(It can be shown that there are 54 lattices having a perfect 6-dimensional cross-section with the same minimum, namely the 53 Baril lattices and 1 Napias lattice. It would be interesting to list the 8-dimensional perfect lattices according to the largest possible dimension m of a perfect cross-section, indeed for $m = 2$ to 7.)
The two results above are classification results. Those we now account for are not.
• Napias ran the Voronoi algorithm, starting from the known perfect lattices with kissing number $s = 36$, keeping the new lattices she found, then ran again the Voronoi algorithm, ... She found 9542 new lattices.
• Finally, Batut determined the Voronoi neighbours of the known perfect lattices having $s = 37$ or $s = 38$, obtaining 146 new lattices.
• All together, there were at the date of July 1st, 1999,

$$1175 + 53 + 9542 + 146 = 10916$$

known 8-dimensional perfect lattices.

In [M], I wrote
The classification for any dimension $n \geq 8$ seems out of scope of the to-day possibilities.
Nevertheless, M. Dutour Sikirić, A. Schürmann and F. Vallentin were able to prove in 2005 *that the list above of* 10916 *perfect, 8-dimensional lattices is actually complete* ([D-S-V]; see also Schürmann's homepage). Their result was obtained by constructing the 8-dimensional Voronoi graph. The main difficulty was caused by the huge number of lattices contiguous to $\mathbb{E}_8$, the Voronoi cone of which has 25 075 566 937 584#2.5 10^{13} facets, and still 83 092 orbits of facets. (They also ran during a few months the Voronoi algorithm in dimension 9. When they stopped the computations, they had found more than 500 000 lattices!)

Here are some more results deduced from the Dutour Sikirić-Schürmann-Vallentin classification.
• All perfect lattices up to dimension 8 have a basis of minimal vectors. This is no longer true in higher dimensions, as shown by the Coxeter lattices (see [M], Sections 5.2 and 5.5) $\mathbb{A}_n^2$ ($n \geq 9$ odd) and $\mathbb{D}_n^+$ ($n \geq 10$ even), which are not generated by their minimal vectors. Note however that bases of minimal vectors for perfect lattices *which are generated by their minimal vectors* might well systematically exist in many dimensions $n \geq 9$.

• All 8-dimensional perfect lattices have a hexagonal section having the same minimum. In particular, they are even in any scale which makes them integral. Note that the same result holds in dimensions 2 to 7 *up to one exception*: $\mathbb{E}_7^*$ is integral when scaled to minimum 3.
(A.-M. Bergé and myself proved ([B-M]) that all perfect, integral lattices having an odd minimum exist in all dimensions $n \geq 10$. The case of dimension 9 is open: our conjecture is that all have a hexagonal section with the same minimum.)
• All 8-dimensional perfect lattices are contiguous to $\mathbb{E}_8$ except two: $\mathbb{A}_8$ (it was proved by Voronoi that $\mathbb{D}_n$ is the only neighbour of $\mathbb{A}_n$ for all $n \geq 3$) and one Napias lattice.
• The *Coxeter conjecture*, according to which $\mathbb{A}_n$ is the less dense of the perfect lattices, is true in dimension 8.
• The Barnes lattice $\mathbb{A}_8^2 = \langle \mathbb{E}_7, \mathbb{A}_8 \rangle$ is the densest perfect 8-dimensional lattice after $\mathbb{E}_8$.
• The only inclusions $L \subset L'$ between perfect lattices having the same minimum in dimension 8 are $[\mathbb{E}_8 : \mathbb{A}_8] = 3$; $[\mathbb{A}_8^2 : A_8] = 2$ (an extension of $[\mathbb{E}_7 : \mathbb{A}_7] = 2$, the only such inclusion in dimensions $n \leq 7$); and $[\mathbb{D}_8 : \mathbb{A}_8] = 2$.

We are going in various WEB pages to comment on the previous lists and to explain how to get the corresponding data.

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2. Laminations and Antilaminations.

We shortly describe various procedures of lamination, as defined in work by Conway and Sloane on the one hand, and by Plesken and Pohst on the other hand, together with the inverse procedure of "antilamination".

1. Let Λ_0 be a Euclidean lattice, of dimension $n_0 \geq 0$ and minimum (= norm) $\min \Lambda = m$. Let $\mathcal{L}_0 = \{\Lambda_0\}$ and define $\mathcal{L}_1 = \mathcal{L}_1(\Lambda_0)$ to be the set of isometry classes of lattices of dimension $n_0 + 1$ and norm m containing a section isometric to Λ_0 whose discriminant is as small as possible. This amounts to saying that the density or that the Hermite invariant of these lattices take the largest possible value among all $n_0 + 1$-dimensional lattices with the same norm m as Λ_0 and which moreover contain a hyperplane section isometric to Λ_0. We can then for each lattice $L \in \mathcal{L}_1$ consider the analogous set $\mathcal{L}_2(L)$ and define $\mathcal{L}_2 = \cup_{L \in \mathcal{L}_1} \mathcal{L}_2(L)$, and finally sets $\mathcal{L}_3, \mathcal{L}_4, \dots$ We obtain in this way a collection of lattices in dimensions n_0, $n_0 + 1$, $n_0 + 2, \dots$ such that each lattice of dimension $n > n_0$ contains with codimension 1 at least one lattice of $\mathcal{L}_{n-n_0-1}$.

Definition 1.1. Lattices which belong to one of the sets $\mathcal{L}_n$, $n \geq n_0$ are called *weakly laminated lattices above* $\mathcal{L}_0$.

2. From dimension $n \geq n_0 + 2$ onwards, there may exist lattices in $\mathcal{L}_n$ with different determinants. We define $\mathcal{L}'_2$ to be the subset of $\mathcal{L}_2$ whose elements are the lattices which have the smallest discriminant. We define similarly $\mathcal{L}'_3$ as the set of lattices of dimension $n_0 + 3$ which are laminated above some lattice of $\mathcal{L}'_2$, and so on. We also set $\mathcal{L}'_0 = \mathcal{L}_0$ and $\mathcal{L}'_1 = \mathcal{L}_1$. We obtain this way a collection of lattices in dimensions n_0, $n_0 + 1$, $n_0 + 2, \dots$.

Definition 2.1. Lattices which belong to one of the sets $\mathcal{L}'_n$, $n \geq n_0$ are called *strongly laminated lattices above* L_0.

Remark 2.2. The determinant and the Hermite invariant of a strongly laminated lattice solely depends on its dimension. However, from dimension $n = n_0 + 3$ onward, there may exist weakly laminated lattices which are denser that the strongly laminated lattices having the same dimension.

Definition 2.3. (Conway and Sloane, [C-S], chapter 6). Strongly laminated lattices above the 0-dimensional lattice to which we give the norm 4 are called *laminated lattices*.

3. Weakly laminated lattices are important from the point of view of perfection. Actually, given L_0 as above,

(1) lattices of a given dimension $n > n_0$ and norm m containing a hyperplane section isometric to Λ_0 on which the determinant attains a local minimum, called *relatively extreme lattices*, can be characterized *à la Voronoi* as lattices which are *relatively perfect* and *relatively eutactic*;
(2) lattices which are perfect relatively to a perfect lattice are perfect in the ordinary sense;
(3) the relatively perfect lattices of dimension $n = n_0 + 1$ are exactly those lattices which possess n independent minimal vectors outside Λ_0.

We refer to [M], chapter XII, section 4 for precise statements and proofs. The fact that an n-dimensional lattice with a perfect hyperplane section of the same norm and n independent minimal vectors outside this section is perfect was used by Barnes in [Barn].

4. In this section, we suppose that Λ_0 is an integral lattice, we set $\mathcal{L}_0^{(a)} = \{\Lambda_0\}$ and we define $\mathcal{L}_1^{(a)}$ to be the set of integral lattices of minimumm $m = \min \Lambda_0$ containing Λ_0 as a hyperplane section whose determinant is as small as possible. We can then construct $\mathcal{L}_2^{(a)}$ starting with the various lattices of $\mathcal{L}_1^{(a)}$ and so on.

Definition 4.1. (Plesken and Pohst, [Pl-P1]). Laminations as above are called *weak arithmetic laminations above* Λ_0, and we define *strong arithmetic laminations above* Λ_0 by restricting from dimension $n_0 + 2$ onward to the smallest possible determinants.
We simply say *arithmetic laminations* (*weak* or *strong*) when we start with the 0-dimensional lattice endowed with the norm 4.

Remark 2.2 applies to arithmetic laminations.

Perfection has in principle nothing to do with arithmetic laminations. However, in practice. it is an interesting source of perfect lattices.

5. We now return to the general notation. Let $\mathcal{L}_{-1} = \mathcal{L}_{-1}(\Lambda_0)$ be the set of hyperplane sections of Λ_0 (if $n_0 > 0$) with the least possible determinant. We can then consider the analogous set for each lattice $L \in \mathcal{L}_{-1}$, obtaining in this way a descending sequence $\mathcal{L}_0 \supset \mathcal{L}_{-1} \supset \mathcal{L}_{-2} \supset \dots$ until we reach dimension 1.

Definition 5.1. The lattices of the above sequence are the *weakly antilaminated lattices under* Λ_0. Lattices of the sequence $\mathcal{L}'_0 \supset \mathcal{L}'_{-1} \supset$

$\mathcal{L}'_{-2} \supset \ldots$ where $\mathcal{L}'_0 = \mathcal{L}_0$, $\mathcal{L}'_{-1} = \mathcal{L}_{-1}$ and the following $\mathcal{L}'_k$ are obtained by restricting at each step to lattices with the least possible discriminant are the *strongly antilaminated lattices under* Λ_0.

Remark 2.2 applies to antilaminations (with n_0+3 replaced by n_0-3). Note that antilamination procedures increase the norm, sometimes strictly. For this reason, we shall not consider "arithmetic" antilaminations.

6. We shall study closely some particularly interesting cases which arise in connection with the lattices Λ_{24} (the *Leech lattice*) and O_{23}. These are the unique unimodular lattices in dimensions 24 and 23 whose minimum (4 and 3 respectively) is the largest possible. The Leech lattice is the unique 24-dimensional strongly laminated lattice, both in the ordinary and arithmetic sense; the lattice O_{23} is the unique 23-dimensional lattice which arises in arithmetic laminations above the 0-lattice endowed with norm 3.

3. Some Perfect Lattices of Odd Minimum.

As one knows since the time of Korkine and Zolotareff, a perfect lattice is proportional to a primitive integral lattice, i.e., such that the scalar products of its vectors are integral and generate $\mathbb{Z}$. In this scale, the lattice is unique up to isometry. In what follows, we systematically rescale perfect lattices as above.

1. An inspection of the list of the 48 perfect lattices of dimension $n \le 7$ shows up 46 even lattices and only 2 odd lattices, namely $\mathbb{Z}$, of minimum 1, and $\sqrt{2}\,\mathbb{E}_7^*$, of minimum 3. The 2005 classification of 8-dimensional perfect lattices by Dutour Sikirić, Schürmann and Vallentin has shown that all these are even. It is proved in [B-M] that odd perefect lattices exist in all dimensions $n \ge 10$; the case of dimension 9 is open.

Note that there exist perfect odd lattices with an even minimum. If Λ is such a lattice, its sub-lattice Λ' which is generated by the set $S(\Lambda)$ of its minimal vectors is even, and the initial lattice is of the form $\langle \Lambda', v_1, \dots, v_k \rangle$ where at least one of the vectors v_i is of odd norm. For instance, for every even $n \ge 10$, the lattice $\sqrt{2}\,\mathbb{D}_n^+$ is perfect with $S(\mathbb{D}_n^+) = S(\mathbb{D}_n)$ and is odd (and integral) for $n \equiv 4 \mod 8$. We will more specially consider (odd) lattices having an odd minimum,

2. The root lattice $\mathbb{A}_n$ is such that $\mathbb{A}_n^*/\mathbb{A}_n$ is cyclic (of order $\det(\mathbb{A}_n) = n+1$); there thus exists for every divisor r of $n+1$ a unique sub-lattice of $\mathbb{A}_n^*$ (*The Coxeter lattice* $\mathbb{A}_n^r$) which contains $\mathbb{A}_n$ to index r. It is perfect except if $r = n+1$ or if $n = 3, 5$ and $r = 2$ (cf. in [M], chapter V, section 2). The case when $r = \frac{n+1}{2}$ (n of course odd) is particularly interesting: the rescaled lattice $\widetilde{\mathbb{A}}_n^r$ is of minimum $n-1$ or $\dfrac{n-1}{2}$ according to whether n is congruent to 1 or to -1 modulo 4 (*loc. cit.*). Consequently, if $n = 7, 11, 15, 19, 23, \dots$, the above lattice is perfect of minimum $3, 5, 7, 9, 11, \dots$. When $n = 7$, this lattice is isometric to $\sqrt{2}\,\mathbb{E}_7^*$. It is well possible that $\widetilde{\mathbb{A}}_{11}^6$ be the perfect lattice of minimum 5 with the smallest possible dimension.

3. Among the known perfect lattices having an odd minimum, two others are of a great interest: they are the two known strongly perfect lattices in dimension 23 whose sets of minimal vectors generate an equiangular family of lines. They have isometric sets of minimal vectors, which generate a lattice L_1 of minimum 5. The second lattice contains the previous one with the index 3, and acquires the minimum 15 when rescaled so as to be integral and primitive. In Venkov's theory of strongly perfect lattices, equiangular families spanned by minimal vectors correspond to strongly perfect lattices with $s = \frac{n(n+1)}{2}$ (the smallest possible value of s for a perfect lattice). By a theorem of Seidel, such families may exist only in dimensions 1, 2 or m where m such that $m+2$ is an odd square. For $m > 2$, the lattice spanned by the minimal vectors has then minimum $m+2$ when rescaled so as to be integral and primitive. Examples are known only for $n = 7$ ($\sqrt{2}\,\mathbb{E}_7^* \sim \mathbb{A}_7^4$) and $n = 23$. Lattices in dimension 23 where constructed by Venkov and Batut in connection with the Leech lattice ([Ven], sections 9 and 19; they are the lattices named F_{23} and F'_{23}).

4. Another important strongly perfect lattice is the unimodular lattice O_{23} of minimum 3, also a projection of a sub-lattice of the Leech lattice. A theorem of Venkov asserts that there are only 5 strongly perfect integral lattices of minimum 3, namely O_{23}, $\sqrt{3}\,\mathbb{Z}$, $\sqrt{2}\,\mathbb{E}_7^*$ and the lattices orthogonal to embeddings in O_{23} of the previous two lattices, named O_{22} and O_{16}.

By antilaminations, we find exactly one lattice in each of the dimensions from 23 to 16. These lattices, denoted O_n, are perfect. For $n = 23, 22, 16$, they are the strongly perfect lattices quoted above. No perfect lattice appears in this way in dimensions $n \leq 15$. Amazingly, we miss O_7 by antilaminations!

There is no reason to restrict to the descending sequence of the densest hyperplane sections. For $n = 22, 21, 20$, these sections are orthogonal to the densest possible lattices in dimension $1, 2, 3$ respectively, for which a Gram matrix is

$$(3)\,, \quad \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)\,, \quad \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}\,.$$

With the matrices (4), $\left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 3 & 0 \\ 0 & 4 \end{smallmatrix}\right)$, we obtain perfect lattices in dimensions 22 and 21 of determinant 4 (instead of 3) and 9 and 12 instead of 8. Several perfect lattices with minimum 3 can be constructed by the above generalization of antilaminations.

By inspection of the list of "weak integral laminated lattices" above O_1 determined by Plesken and Pohst, we find several perfect lattices of minimum 3 (including some of the previous ones) in dimensions 1, 7 (O_1, O_7) and 15 to 23. In particular, this list contains the lattices O_n in the range $n = 23 - 17$. Amazingly, O_{16} is missed by Plesken-Pohst's procedure and their perfect lattice of dimension 15 is missed by the antilamination procedure! In dimensions $23, 22, 21$, this yields $1, 2, 2$

lattices, indeed 5 out of 6 of the lattices described above (the missing one has dimension 21 and determinant 9).

5. Venkov ([Ven], theorem 7.13) has found upper bounds for the invariant s of an integral lattice of minimum 3 and dimension $n \leq 24$, which are sharp for $n \leq 7$, but probably for no larger value of n except $n = 16$ and $n = 23$. Integral lattices of minimum 3 have been considered in detail in Martinet–Venkov's paper [M-V0]. Denote by $s_{\max}(n)$ the maximum value of s in dimension n. It is proved in [M-V0] that $s_{\max}(8) = 30$ and $s_{\max}(9) = 34$, and we believe that $s_{\max}(10) = 40$, $s_{\max}(11) = 52$, and $s_{\max}(12) = 68$. (These values are attained on weak integral laminated of [Pl-P1] and on two other lattices in dimension 10). This would imply that no integral perfect lattice of minimum 3 exist for $2 \leq n \leq 12$ except for $n = 7$. The inequality $s \geq \frac{n(n+1)}{2}$ occurs for $n = 14$, but no perfect lattice is known in this dimension.

4. Structure of the Tables.

The tables consist of Gram matrices written in PARI format together with comments. Each file can be read directly by gp using the usual command \r *name-of-file* (the file must be in the directory in which gp has been opened, unless special directories have been specified in the file *.gprc*).
Thus, a given matrix in usual and gp forms appears as

$$\begin{pmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{pmatrix} \qquad \text{and} \qquad [a1, b1, c1; a2, b2, c2; a3, b3, c3] \quad .$$

However, tables contain

$$foobar = [a1, b1, c1; a2, b2, c2; a3, b3, c3];$$

rather than

$$foobar = [a1, b1, c1; a2, b2, c2; a3, b3, c3] \qquad \text{alone}\,.$$

The semicolon after $[\ldots, c3]$ has been added in order that data should not be displayed on the screen when reading the file. So, to use a particular matrix within a non-gp program, one should remove the semicolon.

A line beginning by \\ is ignored by gp. We have used this possibility to include various comments in the files, which can be read by editing the file (e.g., under *emacs*).

4.0. General notation. All comments which appear in a given file make use of some common notation.

Each Gram matrix has a name which allows gp to recognize it.

With each Gram matrix, we indicate either the perfection rank together with its maximal possible value, namely $\frac{n(n+1)}{2}$, if the lattice is not perfect, or the mention "(perfect)" otherwise. Warning : perfection is not mentionned if the perfection results are stated globally for the whole file.

We systematically give *characteristic invariants* in the form
$[a,\ [b,c],\ [d,e],\ [f_1^{g_1},\dots,f_k^{g_k}]\,]$, where:
$a=\det(\Lambda)$, the determinant of Λ ;
$b=s(\Lambda)$, the number of pairs of minimal vectors of Λ ;
$c=\min\Lambda$, the minimum of Λ ;
$d=s(\Lambda^*)$, the number of pairs of minimal vectors of the dual of Λ ;
$e=\min\Lambda^*$, the norm of the integral primitive scaled copy of Λ^* ;
$[f_1^{g_1},\dots,f_k^{g_k}]\,]$ is the *Smith invariant of* Λ, i.e. the sequence of the elementary divisors for the pair (Λ^*,Λ) ; g_i is omited when it is equal to 1.
[We always assume that Λ is an integral lattice, so that the inclusion $\Lambda\subset\Lambda^*$ always holds.]
The integers k, f_i, g_i satisfy the following properties:
f_1 is the level of Λ, i.e. the annihilator of Λ^*/Λ (and also the smallest of the integers m such that $\sqrt{m}\,\Lambda^*$ is integral);
$f_2\mid f_1,\dots,f_k\mid f_{k-1}$;
$\prod_{i=1}^k f_i^{g_i}=a$;
$\sum_{i=1}^k g_i=n$;
f_k is the g.c.d. of the values of the scalar product $x.y$, $x,y\in\Lambda$.
Note : the formula for the minimum of Λ^* is

$$\min\Lambda^*=\frac{e}{f_1}=\frac{\min\Lambda^*}{|\operatorname{Ann}(\Lambda^*/\Lambda)|}\,.$$

We also give the order of the automorphism group of the lattice, and maybe various remarks: modularity, property as a spherical design, ..., e.g.,, for K_{12}, one will find the comment 3-mod., 5-design.

.../...

4.1. Specific files devoted to perfect lattices.

4.1.1. File *perf2to7.gp*.
This file contains Gram matrices for all perfect lattices in dimensions 2 to 7. Opening *gp* in a directory containing the file, the command $\backslash r$ perf2to7.gp will read the file. Gram matrices are obtained by a command $pn[i]$ where $n = 2, 3, 4, 5, 6$ or 7, and the index i runs through 1 to maximal indices i_n, whose values are $i_2 = i_3 = 1$, $i_4 = 2$, $i_5 = 3$, $i_6 = 7$ and $i_7 = 33$; this table contains **47** lattices.
[For technical reasons, 1×1 matrices are never displayed in our gp-files; nevertheless, there are **48** similarity classes of perfect lattices in dimensions 1 to 7.]

4.1.2. Here is the fundamental modification performed *on April first, 2012:* I have suppressed all the complicated g*p-access* and the tables of LAIHEM, BARIL, NAPIAS (in ten tables), BATUT.

1. One can load the vector *p8.gp* with components the 10916 perfect, 8-dimensional lattices. Then using the command *p8[i]* one gets the *i*th perfect lattice of the vector. Lattices are displayed by decreasing Hermite invariant.

2. By loading *p8d.gp* one obtains SIX vectors p8d7,...,p8d2, where *p8d*i contains the perfect 8-dimensional lattices having a perfect section of the same minimum of dimension i, but none of dimension *i+1* (if $i < 7$).

Thus this section **2** is now reduced to three files, and the former next section **3**, provisionally empty, will contain some data for dimension 9.

4.2. Other families of lattices.

WARNING! Each file **4.2.1**, **4.2.2**, **...** quoted below contains Gram matrices in PARI-GP format. Editing these files (e.g., with emacs) will show the principal invariants of the corresponding lattice as well as the structure of the table if it is divided into several parts.

4.2.1. Lattices contained in the Leech lattice.
File: *Lambda.gp*.
The file contains Gram matrices for the lattices $\Lambda_n^?$ (laminated lattices), K_n, and K'_n and for a few more related lattices (e.g., some arithmetic laminations of Plesken-Pohst), which are all contained in the Leech lattice Λ_{24}.

4.2.2. Sections of the shorter Leech lattice $\mathbf{O_{23}}$.
File: *AntiLaminO23.gp*.
The file contains (1) Gram matrices for the lattices obtained by (weak) antilaminations of O_{23} in dimensions 23 to 8; (2) antilaminations of the

lattice of codimension 2 in O_{23} and determinant 9; (3) a few lattices contained in O_{24}.

4.2.3. Weakly arithmetic laminated lattices of minimum 3.
File: *PlP3.gp*.
The file contains Gram matrices for the lattices obtained by weak laminations of $\sqrt{3}\,\mathbb{Z}$ in dimensions 2 to 23. The data have been taken from Plesken-Pohst' s [Pl-P1] by Huguette Napias. Some of the lattices appeared yet in the previous file.

4.2.4. Some integral lattices of minimum 6.
File: *Min6Lat.gp*.
This file contains lattices related to the duals scaled to minimum 6 of K'_{10} and K'_{18}, and lattices constructed as pull back of $\mathbb{D}_n$ of even binary codes of weighjt 6.

4. Section **4** of the page contains the Voronoi grtaphs in dimensions 2 to 7 and the minimal classes in dimensions 2 to 4.

5. Section **5** of the page is devoted to lattices whose minimal vectors carry the structure of a spherical design of level ≥ 3 (*strongly eutactic lattices*) or ≥ 5 (*strongly perfect lattices*).

6. Section **6** of the page contains various other numerical data.

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