USER'S GUIDE FOR TABLES OF ODD UNIMODULAR LATTICES WITHOUT ROOTS

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1. Introduction

This user's guide is devoted to using *PARI-GP* tables of odd unimodular lattices without roots (i.e., of minimum $m \ge 3$) in dimensions $n \le 28$. These lattices are discrete subgroups of rank n in an n-dimensional Euclidean space E, which may be identified to $\mathbb{R}^n$ once we have chosen an orthonormal basis for E. The tables rely on Bacher-Venkov's work [B-V], where lattices in dimensions $n = 27$ and $n = 28$ are described as *Kneser-neighbours* of $\mathbb{Z}^n \subset \mathbb{R}^n$ relative to a pair (v, p) of a vector $v \in Z^n$ and a prime p.

Recall that given a lattice L, a vector $v \in L$ and an integer $p \ge 2$, the *neighbour of* L *for* (v, p) is

$$L_p^v = \langle L_{p,v}, \tfrac{v}{p} \rangle$$

where

$$L_{p,v} = \{x \in L \mid x \cdot v\} \equiv 0 \mod p\} .$$

It was known before Bacher-Venkov's paper ([C-S], Chapters 16 and 17; [Bo]) that lattices of dimension n and minimum $m \ge 3$ do not exist for $n \le 22$ and that there exists exactly one such odd lattice for $n = 23$ (O_{23}, the shorter Leech lattice), for $n = 24$ (O_{24}, the odd Leech lattice), and $n = 26$ (let's call it O_{26} — Borcherds's lattice, named T_{26} in [Ne-Sl]), but none for $n = 25$. In [B-V], Bacher and Venkov have classified these lattices for $n = 27$ (3 lattices) and $n = 28$ (38 lattices). It turns out that unimodular lattices of minimum 3 in the range $[24, 31]$ share out among two types, related to their parity vectors; see Definition 2 below.

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2. Parity vectors.

Recall that a *parity vector for an integral lattice* L is a vector $v \in L$ such that

$$\forall\, x \in L,\, x \cdot x \equiv v \cdot x \mod 2\,.$$

It is easy to see that parity vectors exist, that they constitute a single class modulo $2L$ if $\det(L)$ is odd, and that there norms $v \cdot v$ are well-defined modulo 8. For unimodular lattices, one has the more precise result: *the norms of parity vectors are congruent to* n *modulo* 8. [The result holds for any signature: replacing L by $L \perp \mathbb{Z}^-$, we are reduced to the case of odd "indefinite lattices", and the result is then easy since these lattices are isometric to a direct orthogonal sum $(\mathbb{Z}^+)^p \perp (\mathbb{Z}^+)^q$; see e.g. [Se], Chapter 5. Note that 0 is a parity vector for L if and only if L is even, so that the dimension of an even unimodular lattice is divisible by 8.]

Definition 1. We denote by $\mathcal{P}$ *the set of parity vectors of* L and by N_{par} the smallest possible norm of a parity vector.

It was proved by Elkies in [El1] that we have $N_{par} \le n$, with equality if and only if $L \simeq \mathbb{Z}^n$, so that if $\min L \ge 2$, we have $N_{par} \le n-8$; and in [El2] that if $\min L \ge 2$ and $n \ge 24$, then $N_{par} \le n - 16$. (For the lattice O_{23}, we have $N_{par} = n - 8$ $(= 15)$.)

In dimension 24, we may have either $N_{par} = n - 24 = 0$, and then L is even, hence isometric to the Leech lattice Λ_{24}, or $N_{par} = n - 16 = 8$, and then L is odd, hence isometric to O_{24}.

In the range $[25, 31]$, N_{par} may *a priori* be equal to $n - 16$ or to $n - 24$.

Definition 2. (Bacher-Venkov.) We say that a unimodular lattice of dimension $n \in [24, 31]$ and minimum 3 is of *general type* (G.T. for short) if $N_{par} = n-16$ and of *exceptional type* (E.T. for short) if $N_{par} = n-24$. We may extend the notion of a G.T. lattice to any n; note ([Ne-V], Corollary 3.4) that n is then bounded from above by 46.

Thus O_{24} and O_{26} (because O_{26} has no vectors of norm 2) are of general type whereas Λ_{24} is of exceptional type.

In dimensions 27 and 28, Bacher and Venkov have proved:

Theorem 3. *Consider unimodular lattices* L *of minimum* 3 *and dimensions* $n = 27, 28$.

(1) *If* $n = 27$, *there are two G.T. lattices and one E.T. lattice.*
(2) *If* $n = 28$, *there are* 36 *G.T. lattices and* 2 *E.T. lattices.*

Notation. In the ordering of [B-V], the G.T. lattices are denoted below by $o_{27a\,i}$, $i = 1, 2$ and $o_{28a\,i}$, $i = 1, \dots, 36$, and the E.T. by o_{27b1}, o_{28b1}, o_{28b2}.

3. Some data.

In this section we restrict ourselves to unimodular lattices L of minimum 3. We denote by L_0 its even sublattice, by S, S_0 and S_0^* the sets of minimal vectors of L, L_0 and L_0^* (the dual of L_0), respectively, and by s, s_0 and s_0^* the corresponding numbers of pairs of minimal vectors.

3.1. **Theta series.** It is proved in [B-V] that the theta series Θ_L of a lattice L as above only depends on its type (general or exceptional). The even coefficients of this series are then those of the even sublattice L_{even} of L, and the theta series also defines that of L_{even}^*.

The proposition below lists the kissing numbers of the three lattices L, L_{even}, and L_{even}^*.

Proposition 4. *The kissing numbers of the lattices above in dimensions* 24 *to* 28 *are as follows:*
- $n = 24$ *(G.T.)* $s = 2048$, $s_0 = 49128$, $s_0^* = 24$ *(*$S_0^* \sim S(\mathbb{Z}^{24})$*)*;
- $n = 26$ *(G.T.)* $s = 1560$, $s_0 = 51090$, $s_0^* = 312$;
- $n = 27$ *(G.T.)* $s = 1332$, $s_0 = 50571$, $s_0^* = 864$;
- $n = 27$ *(E.T.)* $s = 820$, $s_0 = 59787$, $s_0^* = 1$;
- $n = 28$ *(G.T.)* $s = 1120$, $s_0 = 49140$, $s_0^* = 3360$;
- $n = 28$ *(E.T.)* $s = 864$, $s_0 = 53236$, $s_0^* = 1$.

For the sake of completion, we give below the data for $n = 23$:
- $n = 23$ *(*$N_{par} = 15$*)* $s = 2300$, $s_0 = 46575$, $s_0^* = 2300$ *(*$S(L_{even}^*) \sim S(L)$*)*.

3.2. **Strong eutaxy.** Recall (Venkov, [Ve]) that a lattice is *strongly eutactic* if the set of its minimal vectors is a spherical 3-design. This amounts to the fact that the sum of the orthogonal projections to the lines which support its minimal vectors is proportional to the identity, a condition which can be easily checked on a computer. The following proposition is proved (though not explicitly stated) in Section 2 of [Ne-V].

Proposition 5. (Nebe–Venkov). *Let L be a unimodular lattice of minimum* 3 *and dimension $n \in [24, 31]$. If L is of general type, then L, L_0 and L_0^* are strongly eutactic.*

If L is one of the three lattices of exceptional type of dimension27 or 28, none of the lattices L, L_0 and L_0^* is strongly eutactic.

3.3. **Perfection (minimum** 3**).** The basic facts concerning the perfection property can be read in the third chapter of [Ma2]. Recall that the *perfection rank of a lattice* L is the rank $r \in [1, \frac{n(n+1)}{2}]$ in the space of symmetric endomorphisms of E of the set of orthogonal projections

p_x to the minimal vectors x of L. The *co-rank* of L is $\frac{n(n+1)}{2} - r$. A lattice is *perfect* if its perfection rank is maximal.

In [Ve], Venkov has defined the notion of a *strongly perfect lattice*, a lattice the minimal vectors of which constitute a 5-design, and proved that such lattices are indeed extreme, hence in particular perfect. He has also classified the integral, strongly perfect lattices of minimum 3, proving that there are exactly *five* such lattices, one in each of the dimensions 1, 7, 16, 22, and 23 — in dimension 23, this is O_{23}, even a 7-design. For $n \geq 24$, our lattices L are never strongly perfect, and indeed no general perfection rules show up. We list below the status of the lattices L with respect to perfection.

Proposition 6. *Among the unimodular lattices of minimum* 3 *and dimension* $n \in [24, 28]$*, there are* 29 *perfect lattices, all of general type, namely* o_{27a1} *(*$r = 378$*), and* 28 *lattices* $o_{28a\,i}$ *(*$r = 406$*), those with* $i = 1, \dots, 25, 28, 30, 32$*. The remaining* 13 *lattices are listed below according to their perfection co-ranks:*
co-rank 1*:* o_{26}*,* o_{28a29} *; co-rank* 2*:* o_{28a26}*,* o_{28a27}*,* o_{28b37} *;*
co-rank 6*:* o_{27a2}*,* o_{28a31} *; co-rank* 14*:* o_{28a33} *; co-rank* 23*:* o_{24} *;*
co-rank 26*:* o_{27b1} *; co-rank* 37*:* o_{28b38} *; co-rank* 42*:* o_{28a34} *;*
co-rank 70*:* o_{28a35} *; co-rank* 105*:* o_{28a36} *;*
[Note that the perfect lattices above are actually extreme and dual-extreme.]

3.4. **More on parity vectors.** Let Λ be a unimodular lattices of minimum 3, with even sublattice Λ_0, and let v be a parity vector for Λ. We have $\Lambda_0 \subset \Lambda \subset \Lambda_0^*$, and $\frac{v}{2}$ clearly belongs to Λ^*. The easy congruence $N(v) \equiv n \mod 2$ shows that we have $v \in \Lambda \smallsetminus \Lambda_0$ if n is odd, and $v \in \Lambda_0$ if n is even. The quotient Λ_0/Λ_0^*, of order 4, is cyclic, with representatives $\{0, v, \pm\frac{v}{2}\}$ if n is odd, and elementary, with representatives $\{0, w, \frac{v}{2}, w + \frac{v}{2}\}$ if n is even, where w is any vector in $\Lambda \smallsetminus \Lambda_0$. In both cases, we have $L_0^* \smallsetminus L = \{\frac{v}{2}, v \in \mathcal{P}\}$. As a consequence, we have $\min L_0^* = \min(\frac{n-16}{4}, 9)$ ($\frac{n-16}{4}$ for $n = 24, 26, 27$; 9 for $n = 28, 29, 30, 31$). Moreover, except for $n = 28$, $v \mapsto \frac{v}{2}$ induces a one-to-one correspondence between the set of shortest vectors of $\mathcal{P}$ and S_0^*. (For $n = 28$, S_0^* also contains S.)

3.5. **Perfection (minimum** 4**).** All the even sublattices L_0 of the 44 unimodular lattices L of minimum 3 are perfect, so that those for which L is of general type are extreme and dual-extreme (the remaining three lattices have not been tested for eutaxy). Among the lattices L_0^*, those for which $L = \mathrm{O}_{23}$ or L is of general type and dimension 28 are perfect (note that $S(L_o^*)$ then contains $S(L)$), the others are not. These 37 perfect lattices are of course extreme and dual-extreme.

3.6. **Density.** The even lattices L_0 all have minimum 4 and determinant 4. Their Hermite invariants are smaller than those of the laminated lattices for $n = 24$, 26, 30, 31, and are equal to them for for $n = 27$, 28, 29. However they are not the densest known lattices in these dimensions; see just above in this WEB-page the lattices constructed by Roland Bacher.

4. The tables.

The file *unimod23to28.gp* is a PARI-GP-file containing LLL-reduced Gram matrices for the 44 unimodular lattices of minimum 3 which exist in dimensions $n \in [23, 28]$. This can be downloaded in a PARI-GP session (strike $\backslash r$ *unimod23to28.gp*).

Gram matrices are named as above, i.e., o23, o24, o26, 027a1, o27a2, o28a1,. . . ,o28a36 for the G.T. lattices, o27b1, o28b1,o28b2 for the E.T. lattices. In dimensions 27 and 28, one can load them as vectors $o27$ of length 3 and $o28$ of length 38. The G.T. lattices are $o27[i]$, $i = 1, 2$ and $o28[i]$, $i = 1, \ldots, 36$ where the subscript i is the same as in o27ai and o28ai, and the E.T. lattices are similarly $o27[3]$, $o28[37]$, and $o28[38]$.

The file also contains Gram matrices o29 and o31 taken from Nebe=Sloanes catalogue, examples of general type in dimensions 29 and 31, respectively,
and two little gp-programs:

- *esl(a)* outputs an LLL-reduced Gram matrix for the even sublattice of the lattice with Gram matrix a;
- *vpar(a)* outputs a parity vector for the lattice with Gram matrix a.

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