# Perfect and Eutactic Lattices: Extensions of Voronoï's Theory

by

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ABSTRACT. We develop some extensions of the ideas which were introduced by Georges Voronoï in his paper "Nouvelles applications des paramètres continus à la théorie des formes quadratiques : 1 Sur quelques propriétés des formes quadratiques positives parfaites". In particular, we explain fairly wide generalizations of Voronoï's notion of an extreme form (or lattice) for which we can prove a characterization "à la Voronoï" in terms of convenient notions of perfection and eutaxy. We also explain some extensions of the Voronoï algorithm, a problem on which much less is known. To finish, we give a few indications about Venkov's recent work on "strongly perfect" lattices.

**Introduction.** Let $E$ be an $n$-dimensional Euclidean space. By a *lattice of* $E$, we mean a discrete subgroup $\Lambda$ of rank $n$ of $E$. One classically defines (cf. § 2) the *Hermite invariant* $\gamma(\Lambda)$ *of* $\Lambda$, and the *Hermite constant* $\gamma_n = \sup_{\mathrm{rk}\,\Lambda=n} \gamma(\Lambda)$. One of the main problems of the geometry of numbers is to determine $\gamma_n$ and, more precisely, to determine the *critical lattices* (those lattices $\Lambda$ for which $\gamma(\Lambda) = \gamma_n$). We shall consider various extensions of this problem, formerly stated only in terms of real quadratic forms – there is a "dictionary" (see below) which establishes a one-to-one correspondence between similarity classes of lattices and equivalence classes over $\mathbb{Z}$ of positive definite quadratic forms having a given minimum over $\mathbb{Z}$. Though Minkowski's fundamental book "Geometrie der Zahlen" appeared at the end of the last century, it was not before the 1970's that the point of view of lattices became preponderant.

Korkine and Zolotareff wrote three papers on the Hermite constant, solving completely for $n \leq 5$ the above two problems in [K-Z3], published in 1877. Their idea is to determine the lattices on which the local maxima of the Hermite invariant are attained. They call such lattices *extreme* ([K-Z2]), and show that extreme lattices possess the following property (to be called later *perfection* by Voronoï, [V1], **5**): *Every extreme form has at least* $\frac{n(n+1)}{2}$ *representations of its minimum which completely determine this form, if one assumes that this minimum is given* ([K-Z3], **7**, 2°). They classify all perfect lattices of dimension $n \leq 5$ (actually, these lattices are all extreme); their method is of a combinatorial nature: one easily sees that perfect lattices must have at least $\frac{n(n+1)}{2}$ pairs $\pm x$ of minimal vectors, and they prove that this inequality is indeed "almost" sufficient; for a precise statement, see [M], ch. VI, th. 2.1, 4.1, 4.2.

The paper [K-Z3] is the main source of inspiration for the paper [V1] that Voronoï sent thirty years later to "Crelle", containing two important contributions to the development of the ideas of Korkine and Zolotareff:

- A characterisation of extreme lattices (or of extreme forms).
- The introduction of a new algorithm which, starting from a perfect form of a given dimension $n$, produces a finite connected graph containing one form in each equivalence class modulo $\mathrm{GL}_n(\mathbb{Z})$.

Besides these main two problems, we shall also discuss:

• The finiteness problem. Korkine and Zolotareff take for granted the fact that there are (up to similarity) only finitely many perfect lattices; Voronoï proves it ([V1], **7**; cf. also [M], ch. III, th. 5.4).

• The rationality problem. Korkine and Zolotareff ([K-Z3], **7**, 5°) prove that a perfect lattice is rational, i.e. proportional to an integral lattice. Some generalizations yield interesting problems about the field of definition of a lattice.

**2. Notation and dictionary.** For a lattice $\Lambda$ in $E$ (endowed with the scalar product $x.y$ ; we define the *norm of* $x$ as $N(x) = x.x$), the *minimum of* $\Lambda$ is $N(\Lambda) = \min \Lambda = \min_{x\in\Lambda\smallsetminus\{0\}} x.x$ , its *determinant* $\det(\Lambda)$ is the determinant of the *Gram matrix* $\mathrm{Gram}(\mathcal{B}) = (e_i.e_j)$ of some basis $\mathcal{B} = (e_1,\dots,e_n)$ of $\Lambda$ over $\mathbb{Z}$, and the *Hermite invariant of* $\Lambda$ is $\gamma(\Lambda) = \dfrac{N(\Lambda)}{\det(\Lambda)^{1/n}}$. It depends solely on the similarity class of $\Lambda$, and $\gamma(\Lambda)^{n/2}$ is proportional to the density of the sphere packing defined by $\Lambda$.

Similarly, for a positive definite quadratic form $q$ in $\mathbb{R}^n$ with corresponding bilinear form $b$, the *minimum of* $q$ is $\min q = \min_{x\in\mathbb{Z}^n\smallsetminus\{0\}} q(x)$, its *determinant* $\det(q)$ is the determinant of the Gram matrix of $b$ in the canonical basis of $\mathbb{R}^n$, and the *Hermite invariant of* $q$ is defined as above. The *matrix* $A$ *of* $q$ (or *of* $b$) is the matrix $A \in \mathrm{Sym}_n(\mathbb{R})$ such that $q(x) = {}^tXAX$ when $x \in \mathbb{R}^n$ is represented by the column $X$. (We denote by ${}^tM$ the transpose of a matrix $M$.)

Let moreover $S = S(\Lambda) = \{x \in \Lambda \mid N(x) = N(\Lambda)\}$ be the set of minimal vectors of $\Lambda$, and let $s = s(\Lambda) = \frac{1}{2}|S(\Lambda)|$. We define in the same way $S(q)$ and $s(q)$ for a quadratic form $q$.

The dictionary works as follows: given a pair $(\Lambda,\mathcal{B})$ as above, we attach to it the quadratic form $q(x) = N(\sum_i x_ie_i)$, which depends solely on the isometry class of $(\Lambda,\mathcal{B})$. Replacing $\mathcal{B}$ by another basis amounts to replacing $q$ by an equivalent form (modulo $\mathrm{GL}_n(\mathbb{Z})$), whence the correspondence stated in the introduction.

Let us now describe the properties of perfection and eutaxy in both the languages of lattices and quadratic forms. We denote by $\mathrm{End}^s(E)$ the set of symmetric endomorphisms of $E$, i.e. the set of elements $u \in \mathrm{End}(E)$ such that $\forall x, \forall y \in E,\ u(x).y = x.u(y)$.

For $x \neq 0$ in $E$, let $p_x \in \mathrm{End}^s(E)$ be the orthogonal projection on the line $\mathbb{R}x$. For a basis $\mathcal{B} = (e_1,\dots,e_n)$ of $E$, let $\mathcal{B}^* = (e_1^*,\dots,e_n^*)$ be its dual basis (one has $e_i.e_j^* = \delta_{i,j}$). Let $X$ be the column of the components of $x$ in $\mathcal{B}$. Then,

$$\mathrm{Mat}(p_x,\mathcal{B}^*,\mathcal{B}) = \frac{1}{N(x)}\,X\,{}^tX\,.$$

Thus, the following two definitions fit with the dictionary lattices – forms:

The *perfection rank of a lattice* $\Lambda$ is the rank $r$ in $\mathrm{End}^s(E)$ of the $p_x$, $x \in S(\Lambda)$, i.e. the dimension of the span of $\{p_x \mid x \in S(\Lambda)\}$. Similarly, the *perfection rank of a form* $q$ is the rank in $\mathrm{Sym}_n(\mathbb{R})$ of the $X\,{}^tX$, $X \in S(q)$. We say that $\Lambda$ (or $q$) is *perfect* if $r = \frac{n(n+1)}{2}$ $(= \dim \mathrm{End}^s(E) = \dim \mathrm{Sym}_n(\mathbb{R}))$. This definition of perfection is equivalent to Korkine-Zolotareff's definition given in the introduction.

A *eutaxy relation for a lattice* $\Lambda$ is a set of coefficients $\rho_x$, $x \in S(\Lambda)/\{\pm1\}$ such that $\mathrm{Id} = \sum \rho_x p_x$ ; by the dictionary, a *eutaxy relation for a form* $q$ is a set of coefficients $\rho'_x$ such that $A^{-1} = \sum_x \rho'_x X^t X$, for one has

$$A^{-1} = \mathrm{Gram}(\mathcal{B})^{-1} = \mathrm{Mat}(\mathrm{Id}, \mathcal{B}, \mathcal{B}^*)^{-1} = \mathrm{Mat}(\mathrm{Id}, \mathcal{B}^*, \mathcal{B})\,;$$

note that $\dfrac{\rho'_x}{\rho_x} = \dfrac{1}{N(x)}$ is positive.

We say that $\Lambda$ (or $q$) is *weakly eutactic* if there exist some eutaxy coefficients, and *eutactic* if these coefficients are strictly positive. We also use the expressions *semi-eutactic* for the weaker condition $\rho_x \geq 0$, and *strongly eutactic* when all the coefficients are equal, and thus strictly positive, as one sees by considering the trace.

A simple calculation yields the following formulae which characterize the eutaxy coefficients $\rho_x$ and $\rho'_x$: for all $\alpha \in \mathbb{R}^n$, one has

$$(\alpha.\alpha)\,(x.x) = \sum_{x \in S} \rho_x\,(x.\alpha)^2 \quad \text{and} \quad (\alpha.\alpha) = \sum_{x \in S} \rho'_x\,(x.\alpha)^2\,.$$

**3. How to characterize extreme lattices.** The property of eutaxy was introduced by Voronoï to characterize extreme lattices, without giving it a name (the word “eutaxy” appears for the first time in this setting in Coxeter's paper [Cox]).

**3.1. Theorem** (Voronoï). *A lattice is extreme if and only if it is perfect and eutactic.*

For a proof, we refer to [V1], **17** or [M], ch. III, th. 4.6; for numerical data, we refer to [C-S1] and to [M], ch. XIV.

We now explain, following [B-M4], how to generalize this theorem to some important families of lattices. For the proofs, we refer to [M], ch. X. A wide generalization (though leaving aside some interesting examples, such as lattices with a given section, cf. [M]. ch. XII) is obtained by taking families $\mathcal{F}$ of the following type: we consider a closed subgroup $\mathcal{G}$ of $\mathrm{GL}(E)$ and a lattice $\Lambda_0$ of $E$, and we take for $\mathcal{F}$ the orbit of $\Lambda_0$ under $\mathcal{G}$. For convenient choices of $\mathcal{G}$ and of $\Lambda_0$, we obtain useful families for which we are able to prove a theorem of the kind of 3.1. The idea is to make use of the tangent space $\mathcal{T}_0 \subset \mathrm{End}(E)$ at the unit of $\mathcal{G}$ (which is a Lie group, since it is closed in $\mathrm{GL}(E)$), and more preciseley of its image $\mathcal{T}$ by $v \mapsto v + {}^t v$ in $\mathrm{End}^s(E)$, and to define convenient notions of perfection and of eutaxy with respect to $\mathcal{T}$.

First, we define the *Voronoï scalar product* on $\mathrm{End}^s(E)$ by $\langle u, v\rangle = \mathrm{Tr}(v \circ u)$ (a similar definition applies to $\mathrm{Sym}_n(\mathbb{R})$). Moreover, given a subspace $H$ of $\mathrm{End}^s(E)$, we denote by $p_H$ the corresponding orthogonal projection on $H$ in $\mathrm{End}^s(E)$. For $x$ non-zero in $E$, let $\omega_x = p_H(p_x)$ and let $\Omega = p_H(\mathrm{Id})$; these are elements of $H$.

**3.2. Definition.** Let $H$ be a subspace of $\mathrm{End}^s(E)$. We say that a lattice $\Lambda$ is *$H$-perfect* if $H$ is spanned by the $\omega_x$, $x \in S(\Lambda)$ and that it is *$H$-eutactic* if there is a relation $\Omega = \sum_{x \in S(\Lambda)} \rho_x \omega_x$ with strictly positive coefficients $\rho_x$.

We then introduce a more precise notion of extremality:

**3.3. Definition.** We say that a lattice $\Lambda \in \mathcal{F}$ is *$\mathcal{F}$-extreme* if the Hermite invariant is a local maximum at $\Lambda$ among lattices of $\mathcal{F}$, and *strictly $\mathcal{F}$-extreme* if there is a neighbourhood $\mathcal{V}$ of $\Lambda$ in $\mathcal{F}$ such that the strict inequality $\gamma(\Lambda') < \gamma(\Lambda)$ holds for every $\Lambda' \in \mathcal{V}$ which is not similar to $\Lambda$.

**3.4. Theorem** (A.-M. Bergé, J. M., [B-M4]). *With the above notation, assume that $\mathcal{F}$ is invariant under homotheties and transposition. Then:*

1. *A lattice $\Lambda \in \mathcal{F}$ is strictly $\mathcal{F}$-extreme if and only if it is $\mathcal{T}$-perfect and $\mathcal{T}$-eutactic.*
2. *If $\Lambda$ is $\mathcal{F}$-extreme but not strictly $\mathcal{F}$-extreme, there is a path $\varphi : [0,1] \to \mathcal{F}$ such that (i) $\varphi(0) = \Lambda$, (ii) $\varphi(t)$ and $\varphi(t')$ are not similar for $t' \neq t$, and (iii) for $t > 0$, $S(\varphi(t))$ is a fixed set which generates a proper subspace of $E$.*

This theorem applies to the family of all lattices, with $\mathcal{G} = \mathrm{GL}(E)$. It implies Voronoï's theorem, and shows moreover that extreme lattices are indeed strictly extreme, a result which is implicit in [K-Z3].

In most of the applications, we first prove that the minimal vectors of extreme $\mathcal{F}$-lattices span $E$; condition (2) of the above theorem then shows that $\mathcal{F}$-extreme $\mathcal{F}$-lattices are indeed strictly $\mathcal{F}$-extreme.

We now give three examples, all of which were found before theorem 3.4 was proved.

**3.5. Example. G-lattices.** Let $G \subset \mathrm{O}(E)$ be a finite subgroup of the orthogonal group $\mathrm{O}(E)$. We say that a lattice $\Lambda$ is a *$G$-lattice* if it is invariant under the action of $G$. Note that this notion depends not only on the structure of the group $G$, but also on the representation $\rho : G \to \mathrm{O}(E)$ afforded by the inclusion $G \subset \mathrm{O}(E)$. Let $\mathcal{F}$ be the set of $G$-lattices. A necessary condition for $\mathcal{F}$ not to be empty is that $\rho$ should be rational over $\mathbb{Q}$; the converse is true, for, given a basis $(e_1, \dots, e_n)$ of $E$, the vectors $se_i$, $s \in G$ span a lattice $\Lambda_0 \in \mathcal{F}$.

Let $\mathcal{G} = \{u \in \mathrm{GL}(E) \mid \forall s \in G,\ su = us\}$ ($\mathcal{G}$ is the group of invertible elements of the commuting algebra $\mathrm{End}_G$ of $\rho$), and let $\mathrm{End}^s_G(E) = \mathrm{End}_G(E) \cap \mathrm{End}^s(E)$. Theorem 3.4 applies with $\mathcal{F}$, $\mathcal{G}$, $\Lambda_0$ as above and $\mathcal{T} = \mathrm{End}^s_G(E)$. One has

$$\omega_x = \frac{1}{|G|} \sum_{s \in G} s p_x s^{-1} = \frac{1}{|G|} \sum_{s \in G} p_{sx} \quad \text{and} \quad \Omega = \mathrm{Id}\ .$$

We say $G$-extreme, $G$-perfect and $G$-eutactic rather than $\mathcal{F}$-extreme, $\mathcal{F}$-perfect, $\mathcal{F}$-eutactic. It is now easy to make theorem 3.4 explicit, and we actually obtain:

**3.5'. Theorem** ([B-M2]). *For a $G$-lattice $\Lambda$, the following conditions are equivalent:*

1. *$\Lambda$ is $G$-extreme.*
2. *$\Lambda$ is strictly $G$-extreme.*
3. *$\Lambda$ is $G$-perfect and $G$-eutactic.*

When $G = \{1\}$, the above theorem reduces to Voronoï's theorem 3.1.

**3.6. Example. Duality.** We consider the following invariant

$$\gamma'(\Lambda) = (\gamma(\Lambda)\gamma(\Lambda^*))^{1/2} = (N(\Lambda)N(\Lambda^*))^{1/2}$$

introduced in [B-M1], and we say that $\Lambda$ is *(strictly) dual-extreme* if $\gamma'$ attains on $\Lambda$ a (strict modulo similarity) maximum. This invariant has an interpretation in the setting of theorem 3.4, but one must replace $E$ by $E \times E$, and consider the local maxima of $\gamma$ on the family $\mathcal{F} = \{(\Lambda, \Lambda^*) \mid \Lambda \in E\}$. The theory of theorem 3.4 applies with

$$\mathcal{G} = \{(u, {}^t u^{-1}) \in \mathrm{End}^s(E) \times \mathrm{End}^s(E) \subset \mathrm{End}^s(E \times E)\}\,.$$

We say that $\Lambda$ is *dual-perfect* if the set of $p_x, p_y$, $x \in S(\Lambda), y \in S(\Lambda^*)$ spans $\mathrm{End}^s(E)$ and *dual-eutactic* if there is a relation

$$\sum_{x \in S(\Lambda)} \rho_x\, p_x = \sum_{x \in S(\Lambda^*)} \rho^*_y\, p_y$$

with strictly positive coefficients $\rho_x, \rho^*_y$. Theorem 3.4 takes then the following form:

**3.6'. Theorem** ([B-M1]). *For a lattice $\Lambda$, the following conditions are equivalent:*

(1) $\Lambda$ *is dual-extreme.*
(2) $\Lambda$ *is strictly dual-extreme.*
(3) $\Lambda$ *is dual-perfect and dual-eutactic.*

Note that the critical lattices for the constant $\gamma'_n$ are known only for $n = 1, 2, 3, 4, 8$.

**3.7. Example. Isoduality.** We come back to the space $E$ itself. Let $\sigma \in \mathrm{O}(E)$. We say that $\Lambda$ is *$\sigma$-isodual* if $\sigma$ is an isometry of $\Lambda$ onto $\Lambda^*$ (or of $\Lambda^*$ onto $\Lambda$, it amounts to the same). The theory of theorem 3.4 works for this family $\mathcal{F}_\sigma$ of $\sigma$-isodual lattices, with

$$\mathcal{G} = \mathcal{G}_\sigma = \{u \in \mathrm{End}(E) \mid {}^t u \sigma u = \sigma\} \;\text{ and }\; \mathcal{T} = \mathcal{T}_\sigma = \{v \in \mathrm{End}^s(E) \mid v\sigma = -\sigma v\}\,.$$

However, *extreme $\sigma$-lattices* need not be strictly extreme in $\mathcal{F}_\sigma$. It is nevertheless the case for two important families. We say that a lattice is *orthogonal* (resp. *symplectic*) if $\sigma^2 = \mathrm{Id}$ (resp. if $\sigma^2 = -\mathrm{Id}$); this amounts to saying that the bilinear form $b : (x, y) \mapsto x.\sigma y$ is symmetric (resp. alternating). For these families, one can prove:

**3.7'. Theorem** ([B-M4]). *For a $\sigma$-isodual lattice $\Lambda$, with $\sigma$ orthogonal or symplectic, the following conditions are equivalent:*

(1) $\Lambda$ *is $\sigma$-extreme.*
(2) $\Lambda$ *is strictly $\sigma$-extreme.*
(3) $\Lambda$ *is $\sigma$-perfect and $\sigma$-eutactic.*

The case of symplectic lattices is of great importance: the existence of $\sigma$ with $\sigma^2 = -\mathrm{Id}$ yields a complex structure of $E$ (thus, $n$ must be even), and symplectic lattices modulo symplectic isometries (those which commute with $\sigma$) are in one-to-one correspondence with isomorphism classes of principally polarized abelian varieties.
[Actually, the imaginary part of $b$ (for $i = \sigma$) is unimodular-alternating; conversely, starting as usual from a Riemann form $H$, we recover the Euclidean space $E$ by making use of the real part of $H$.]

Examples of isodual lattices with a large Hermite invariant have been given by Conway and Sloane in [C-S2]. Actually, their lattices are symplectic when their rank is even, and orthogonal otherwise. Note that the maxima of $\gamma$ on isodual lattices are known only for the dimensions $1, 2, 3, 4, 8$.

A generalization of theorem 3.4 in the setting of differential geometry (concerning the "systole" of a Riemannian manifold) has been recently obtained by Bavard. It is not possible to explain his theorem in this short report, for which we refer the reader to [Bav1]; we shall simply remark that the usual Hermite invariant has an interpretation as "the systole of a flat torus".

**4. The Voronoï algorithm.** In this section, we shall follow tradition, and work with quadratic forms rather than with lattices. From an algorithmic point of view, it is the better choice. However, the geometric description with lattices may be useful; Voronoï himself uses such an approach (without saying it) in his study of the forms attached to the lattice $\mathbb{D}_n$, which are described in relation to $\mathbb{Z}^n$ ([V1]; see in particular the first displayed formula in **38**).

Let $q$ be a positive definite quadratic form with minimum $m$ and matrix $A$. We identify the space of quadratic forms with $\mathrm{Sym}_n$. The *Voronoï domain of* $q$ is the convex hull $\mathcal{D} = \mathcal{D}_q$ of the rays $\lambda X^tX$, $\lambda \geq 0$, $X \in S(q)$ ; it is a convex polyhedral cone. From the definition, we see that $q$ is perfect if and only if the interior of $\mathcal{D}$ is not empty, and that $q$ is semi-eutactic (resp. eutactic) if and only if $A^{-1}$ belongs to $\mathcal{D}$ (resp. to the relative interior of $\mathcal{D}$).

The Voronoï domain is bounded by facets which are hyperplanes in the subspace of $\mathrm{Sym}_n$ spanned by $\mathcal{D}$. Given such a facet $\mathcal{F}$, we say that a non-zero form (or symmetric matrix $F$) is a *face vector for* $\mathcal{F}$ if it is orthogonal in $\mathrm{Sym}_n$ (endowed with the Voronoï scalar product $\mathrm{Tr}(MM')$) to all vectors of $\mathcal{F}$.

For $\theta \in \mathbb{R}$, $F \in \mathrm{Sym}_n$ and $x \in \mathbb{R}^n$ with corresponding column $X$, one has

$$^tX(A+\theta F)X = \langle A+\theta F, X^tX\rangle = \langle A, X^tX\rangle + \theta\langle F, X^tX\rangle = q(x) + \theta\langle F, X^tX\rangle\,.$$

Now, under a small enough transformation of a quadratic form, all minimal vectors of the new form are minimal vectors of the old one. Hence, for $\theta$ small enough, $q$ and the form $q_\theta$ corresponding to $A+\theta F$ have the same minimal vectors if and only if $\langle F, X^tX\rangle = 0$ for all $X \in S(q)$.

We now distinguish two cases:

<u>case 1</u>. $q$ is not perfect.
We can choose an $F$ orthogonal to all $X^tX$, $X \in S(q)$. It can then be shown ([V1], **3 - 5**; [M], ch. VII, prop. 2.6) that there exist $\rho_1, \rho_2 > 0$ such that $S(q_\theta) = S(q)$ for $-\rho_1 < \theta < \rho_2$ and $\min q_\theta < \min q$ for $\theta < -\rho_1$ or $\theta > \rho_2$ ; then, $q_{-\rho_1}$, $q_{\rho_2}$ have the same minimum as $q$, but a strictly larger set of minimal vectors. After a finite number of steps, one obtains a perfect form.
[Refinement: let $r$ be the perfection rank of $q$; then ([M], ch. IX, th. 1.9, (4)), one can proceed in such a way that $q_{\rho_1}$ and $q_{\rho_2}$ have perfection rank $r+1$.]

<u>case 2</u>. $q$ is perfect.
For each facet $\mathcal{F}$ of $\mathcal{D}_q$, the corresponding face vectors are proportional. We can then choose a face vector $F$ in such a way that there exists $\rho > 0$ with the following two properties:

(1) $\min q_\theta < \min q$ for $\theta < 0$ or $\theta > \rho$ ;
(2) $S(q_\theta)$ is the set of minimal vectors $X$ of $q$ with $X^tX \in \mathcal{F}$ for $0 < \theta < \rho$.

Then ([V1], **3.**; [M], ch. VII, § 2), $q_\rho$ is again a perfect form.

**4.1 Definition.** The form $q_\rho$ defined above is called the *neighbour form of* $q$ *with respect to* $\mathcal{F}$ (or *along* $F$).

The neighbouring relation gives the set of perfect forms of dimension $n$ and given minimum $m$ the structure of a graph. Clearly, the neighbour forms corresponding to faces of $q$ which are equivalent under $\mathrm{Aut}(q)$ are equivalent; thus, we can define a quotient graph, which is finite. The fundamental result of Voronoï is:

**4.2. Theorem.** *The neighbouring graph is connected (and finite modulo equivalence).*

Voronoï made some calculations of neighbour forms, and found in particular the complete structure of the neighbouring graph in dimension $n \leq 5$, obtaining in this way a new verification of Korkine-Zolotareff's classification up to dimension 5.

He also began to study the neighbouring graph in dimension 6, as one sees from the following remark he made in the introduction of [V1] (p. 100): *Ce n'est qu'à partir des formes positives à six variables que j'ai rencontré des formes quadratiques positives qui jouissent de la propriété (I)* (i.e., are perfect) *et ne sont pas des formes extrêmes.* However, he had not enough time to finish his research; the classification in dimension 6 was completed half a century later by Barnes in his difficult paper [Bar], where he used Voronoï's method, and made all calculation by hand. He proved that there exist seven perfect 6-dimensional lattices, among which exactly one is not extreme, in conformity with the above assertion of Voronoï.

The graph for 7-dimensional forms was obtained by Jaquet in his thesis ([J], [J1]). He had to develop new algorithms to be able to list all the facets of perfect forms and classify them up to equivalence ($\mathbb{E}_7$ has more than 70,000,000 facets!) and to write a heavy computer program. He proved that there are exactly 33 perfect 7-dimensional lattices, confirming previous calculations due to K. Stacey, relying on earlier work by Watson extending [W1], without any connection with the Voronoï algorithm.

Computations done by H. Napias (Ph. D. thesis, Bordeaux, 1996) produced more than 10,700 inequivalent perfect forms in dimension 8, among which more than 6000 appear as neighbours of $\mathbb{E}_8$. This shows clearly that the determination of the neighbouring graph in dimension 8 lies far beyond the capabilities of the existing computers.

Nevertheless, it is possible to develop the Voronoï algorithm in new directions. The formal generalization deals with a subspace $\mathcal{T}$ of $\mathrm{Sym}_n$. We define the $\mathcal{T}$-domain of Voronoï of a $\mathcal{T}$-perfect form to be the orthogonal projection on $\mathcal{T}$ (for the Voronoï scalar product on $\mathrm{Sym}_n$) of the usual Voronoï domain. We choose a fixed form $q_0$, and then restrict ourselves to those forms $q$ such that $q - q_0$ belongs to $\mathcal{T}$. Neighbour forms are defined in the same way. However, it may happen that some faces have no neighbour, but, despite this possible existence of some "culs-de-sac", a Voronoï-like theorem could be obtained in some interesting settings. We refer to [M], ch. XIII, §§ 1,2 for the details of the general theory.

There is no reason to develop a detailed "$\mathcal{T}$-Voronoï" theory for arbitrary subspaces $\mathcal{T}$. Here are two examples which yield interesting applications.

**4.3. Example. G-lattices again.** The notation is that of example 3.5; in particular, $\mathcal{T} = \mathrm{End}^s_G$. We shall speak indifferently of $G$-lattices or of $G$-forms.

**4.3'. Theorem** ([B-M-S]). *The connected components of the Voronoï neighbouring graph for G-perfect G-lattices are in one-to-one correspondence with the isomorphism classes of integral representations afforded by G-lattices.*

This extended algorithm was used by Sigrist (work in progress) for cyclotomic lattices, attached to a cyclic group $G$ of order $\ell$ whose generators have $\phi_\ell$ as minimal polynomial. In particular, he found all $G$-perfect lattices when $\ell$ is one of the prime numbers $3, 5, 7, 11, 13, 17$, producing respectively $1, 1, 2, 5, 25, 1344$ $G$-perfect forms. One can then extract from his lists of perfect $G$-lattices those for which the Hermite invariant attains the greatest value. As a consequence, we derive the optimal upper bound of this invariant for lattices whose automorphism group contains a group $G$ as above.

**4.4. Example. Lattices with a given minimal section.** We consider a lattice $\Lambda_0$ in a subspace $F$ of $E$ of dimension $n_0 < n$, and the family $\mathcal{E}$ of those lattices $\Lambda$ such that $\Lambda \cap F = \Lambda_0$ and $\min \Lambda = \min \Lambda_0$. The $\mathcal{E}$-extreme lattices can be characterized by properties of perfection and eutaxy relative to $\Lambda_0$ (a fact mentioned in § 3, cf. [M], ch. XII). But there is also a Voronoï algorithm relative to $F$ (cf. [M], ch. XIII). Such algorithms were implemented by M. Laïhem and J.-L. Baril (Ph. D. theses, Bordeaux, 1992 and 1996)). As a consequence, we now know in particular that there are exactly 1175 perfect 8-dimensional lattices possessing a 7-dimensional perfect section with the same norm and 53 more perfect lattices which are direct sums of a perfect 6-dimensional lattice and a $\mathbb{A}_2$-lattice normalized to the same minimum. (This last result needs a modification of the above theory, cf. [M], ch. 13, § 6.)

The usual Voronoï algorithm itself has important applications. With the mere knowledge of Voronoï's and Barnes' results for dimensions $n \leq 6$, Watson obtained in [W2] sharp estimates of the invariant $s$ up to dimension 9, for instance $s \leq 75$ for non-perfect lattices of dimension 8, and the optimal bound $s \leq 136$ for dimension 9.

It would be interesting to look again at Watson's methods, taking into account recent progress of the theory of perfect lattices (and of computer science).

**5. Eutaxy and cell decomposition.** The following relation among lattices of $E$:

$$\Lambda \sim \Lambda' \iff \exists u \in \mathrm{GL}(E), u(\Lambda) = \Lambda' \text{ and } u(S(\Lambda)) = S(\Lambda')$$

is an equivalence relation, for which two similar lattices are equivalent. We call *cells* the equivalence classes *on the set of lattices modulo similarity* (rather than *minimal classes* as in [B-M3] and [M], though they are only quotients by finite groups of cells in the sense of topology, see below).

The perfection rank $r$ is constant on a cell and is equal to its codimension in $\mathrm{End}^s(E)$. Cells of dimension 0 (resp. 1) are reduced to the similarity class of a single perfect lattice (resp. of a Voronoï neighbouring path connecting two perfect lattices).

It is important to consider the description of cells in terms of quadratic forms, where they appear as the interior of $r$-dimensional convex polyedra in $\mathbb{R}^{n(n+1)/2}$ whose summits are perfect forms.

We only consider in the sequel the cells built with *well rounded* lattices or forms (those which possess $n$ independent minimal vectors; the terminology is Ash's). (The complete theory easily reduces to this particular case.)

Actually, the set of all well rounded positive definite quadratic forms with a given minimum has a structure of an infinite cell complex; our set of classes is its finite quotient by $\mathrm{GL}_n(\mathbb{Z})$, but the classes may be no more topological cells.

In practice, one obtains a finite cell complex by keeping sometimes several representative of a given class. For instance, for $n = 4$, the Voronoï graph should be written $\mathbb{A}_4$ ——— $\mathbb{D}_4$ ——— $\mathbb{D}_4$ with two copies of $\mathbb{D}_4$ in order that the second 1-dimensional cell should not be folded; then, several copies of 1-dimensional cells should be used as edges for 2-dimensional cells, and so on.

The cell named $b_8$ in [B-M1] is then represented by a square with summits equivalent to $\mathbb{A}_4, \mathbb{D}_4, \mathbb{A}_4, \mathbb{D}_4$ whose 4 edges are equivalent to the path $\mathbb{A}_4 - \mathbb{D}_4$, whereas the cell $a_8$ corresponds to a triangle with summits equivalent to $\mathbb{D}_4, \mathbb{D}_4, \mathbb{A}_4$ with 2 edges equivalent to the path $\mathbb{A}_4 - \mathbb{D}_4$ and 1 edge equivalent to the path $\mathbb{D}_4 - \mathbb{D}_4$. To describe the 3-dimensional cells, one must then repeat some 2-dimensional cells, and thus consider several new copies of the 0-dimensional cells $\mathbb{A}_4$ and $\mathbb{D}_4$.

We refer the reader to [Bt] for a concrete description in dimension 5. However, it should be noticed that for algorithmic reasons, Batut starts with 0-dimensional cells, whereas the cell complex described above is better understood in each dimension by determining all cells which appear as edges of a given cell.

This cell decomposition of the space of lattices modulo similarity was first considered by Štogrin in [St], who stated that there are only finitely many cells, found the classification for dimensions $2, 3, 4$ and proved in general by an argument of convexity that the minimum on a cell of the Hermite invariant $\gamma$, if any, is attained on a single lattice or form. (Mahler's compactness lemma shows that the minimum of $\gamma$ exists on the closure of each well rounded cell.) A much more developed paper is Avner Ash's [Ash2].

These results were rediscovered by Bergé and Martinet (who were not aware of [St]) in [B-M3], with however an important complement: they introduced the notion of a *weakly* eutactic lattice (see § 2), and proved that weakly eutactic lattices in a cell are exactly those lattices where $\gamma$ attains its minimum, see [M], ch. IX for the details and some further results. This proves the finiteness up to similarity of the set of weakly eutactic lattices of a given dimension; in particular, there are only finitely many eutactic lattices, a result proved previously by Avner Ash ([Ash1]) in connection with topological Morse theory, and also, as was known to Voronoï, finitely many perfect lattices, since all perfect lattices are weakly eutactic.

The classifications of well rounded cells and (weakly) eutactic lattices were found by the autors quoted above when $n = 2, 3, 4$, where one has respectively $3, 5, 18$ well rounded cells. The difficult 5-dimensional classifications were done recently by Batut ([Bt]), who found in particular 136 well rounded cells.

The cell classification appears to be a wide generalization of Voronoï's finiteness theorem and of the Voronoï neighbouring graph, which consists in the union of 0- and 1-dimensional cells. Moreover, it extends naturally to other objects, e.g. $G$-lattices, pairs $(\Lambda, \Lambda^*)$ of lattices, orthogonal or symplectic lattices, though no Voronoï algorithm is in general available.

Finiteness theorems have been proved for the $\mathcal{F}$-extreme (or even sometimes for the
$\mathcal{F}$-perfect) lattices of the above list. Perfect $G$-lattices are moreover rational, but rationality is not a general property. It is known that these $\mathcal{F}$-extreme lattices are algebraic, and, actually, non-rational $\mathcal{F}$-extreme lattices exist for orthogonal or symplectic lattices. (Whether there exist non-rational extreme pairs $(\Lambda, \Lambda^*)$ is not known.)

Similarly, weakly eutactic lattices are algebraic, and Batut has found a 5-dimensional eutactic lattice whose field of definition is of degree 9 over $\mathbb{Q}$. The proofs of algebraicity rely on an argument of real algebraic geometry, modelled on the argument due to Bergé and Matignon given in [Ber] for the pairs $(\Lambda, \Lambda^*)$.

No general Voronoï algorithm is known (examples 4.3 and 4.4 concern affine families). However, a remarkable example has been obtained very recently by Bavard; it concerns certain families of symplectic lattices which can be parametrized by the upper half plane thanks to the interpretation of symplectic lattices as principally polarized abelian varieties and their parametrization by Siegel's space. Bavard uses geodesics of the upper half plane to construct his graph, and proves that it is connected. We refer the reader to [Bav2] for the highly technical details involved in this paper.

**6. Strong perfection and spherical designs.** We explain shortly in this last section a theory of Boris Venkov ([Ven]). We define a highly restrictive notion of extremality, which nevertheless applies to important classes of lattices, for instance to the class of 32-dimensional even unimodular lattices.

Let $t$ be a positive integer. A *spherical $t$-design* is a finite subset $S$ of some sphere $\mathbb{S} \subset E$ with center 0 such that the equality

$$\int_{\mathbb{S}} f\, d\sigma = \frac{1}{|S|} \sum_{x \in S} f(x)$$

holds for all polynomials $f$ of degree at most $t$ ($d\sigma$ is the usual measure on $\mathbb{S}$ normalized by $\int_{\mathbb{S}} d\sigma = 1$). This amounts to saying that the equality $\sum_{x \in S} f(x) = 0$ holds for all non-constant harmonic polynomials of degree at most $t$. A $t$-design is a $t'$-design for all $t' \leq t$.

We shall use the above definition when $S$ is the set $S(\Lambda)$ of minimal vectors of a lattice $\Lambda \subset E$. Note that, since $S$ is then symmetric, every $2t$-design is also a $2t+1$-design. We prove easily:

**6.1. Proposition.** *Let $t = 2m+1$. Then, the set $S$ of minimal vectors of a lattice $\Lambda$ is a spherical $t$-design if and only if there exists $c \in \mathbb{R}$ such that the following identity*

$$\sum_{s \in S} (x.\alpha)^{2m} = c(\alpha.\alpha)^m$$

*holds for all $\alpha \in E$.*

**6.2. Definition.** We say that a lattice $\Lambda$ is *strongly perfect* if $S(\Lambda)$ is a spherical 5-design.

By proposition 6.1, $S(\Lambda)$ is a 3-design if and only if $\Lambda$ is strongly eutactic (i.e., eutactic with equal coefficients, see § 2). For 5-designs, one proves:

**6.3. Proposition.** *A strongly perfect lattice is extreme.*

Venkov has obtained some classification results; for small dimensions, one has:

**6.4. Theorem** (Venkov). *A lattice of dimension $n \leq 9$ is strongly perfect if and only if it is similar to one of the lattices $\mathbb{Z}$, $\mathbb{A}_2$, $\mathbb{D}_4$, $\mathbb{E}_6$, $\mathbb{E}_6^*$, $\mathbb{E}_7$, $\mathbb{E}_7^*$ or $\mathbb{E}_8$.*

Other results of Venkov concern dimension 11 (there does not exist any strongly perfect lattice), integral lattices of minimum 3 (there exist exactly five strongly perfect lattices, one in each of the dimensions $1, 7, 16, 22, 23$), and lattices with $s = \frac{n(n+1)}{2}$ (the configuration of their minimal vectors defines an equiangular family of lines).

Note also the following general property: *the Bergé-Martinet invariant $\gamma'(\Lambda)$ of a strongly perfect lattice satisfies the inequality $\gamma'(\Lambda)^2 \geq \frac{n+2}{3}$.*

To finish, we describe some results concerning even unimodular lattices (integral lattices of determinant 1 whose vectors have even norms). Hecke proved that such lattices exist if and only if $n$ is divisible by 8, and must have minimum $m \leq 2\left\lfloor \frac{n}{24} \right\rfloor + 2$. Those for which equality holds are called *extremal.* (Examples: $\mathbb{E}_8$, and the Leech lattice $\Lambda_{24}$.) By applying the theory of modular forms and making use of theta series with spherical coefficients, Venkov proved the following theorem:

**6.5. Theorem.** *Let $\Lambda$ be an even unimodular lattice of dimension $n$. If $n \equiv 0$ (resp. $n \equiv 8$, resp. $n \equiv 16$) modulo 24, then $S(\Lambda)$ is a 11- (resp. 7-, resp. 3-) spherical design. In particular, extremal even unimodular lattices of dimension $n \equiv 0, 8 \mod 24$ are strongly perfect.*

This theorem produces examples of strongly perfect lattices in dimensions 8, 24, 32, 48, 56 and 80 (two examples recently found by Bachoc and Nebe; whether there exist 72-dimensional extremal lattices is one of the main open problems of the theory of unimodular lattices).

The theory partially extends to lattices $\Lambda$ of level $\ell$ (even lattices $\Lambda$ such that $\sqrt{\ell}\,\Lambda^*$ is again an even lattice) for which $\Theta_\Lambda = \Theta_{\sqrt{\ell}\Lambda^*}$, and in particular to $\ell$-modular lattices in the sense of Quebbemann (even integral lattices $\Lambda$ such that there exists a similarity $\sigma : \Lambda^* \to \Lambda$ of ratio $\sqrt{\ell}$; 1-modular is unimodular). For $\ell = 2, 3$, one can prove results similar to theorem 6.5. In particular, this applies to the Coxeter-Todd lattice $K_{12}$ (with $\ell = 3$), to the Barnes-Wall lattice $\Lambda_{16}$ ($\ell = 2$) and to the 2-modular lattices of dimension 32 and minimum 6, four examples of which are known, found by Quebbemann, Quebbemann, Bachoc and Nebe.
[There also exist in the same genus (Elkies, Nebe) lattices which *are not* 2-modular.]

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