# EXTENSIONS OF THE PERFECT LATTICE $\mathbf{P_6^6}$

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Abstract. We construct two families $L_n, L'_n$ of perfect lattices which extend beyond dimension 6 the six-dimensional perfect lattice $P_6^6$, with $L'_n \simeq L_n$ if $n$ is even, but not if $n$ is odd.

## Introduction

We denote by $E$ a Euclidean space, of dimension $n \geq 2$. Given a lattice $\Lambda \subset E$, its *minimum* is $\min \Lambda = \min_{x \in \Lambda \setminus \{0\}} x \cdot x$. We denote by $S(\Lambda)$ its set of *minimal vectors* (those with $x \cdot x = \min \Lambda$) and set $s(\Lambda) = \frac{1}{2} |S(\Lambda|$.

We refer to [M], in particular to chapters 1 and 3 for general definitions relative to (Euclidean) lattices, including the perfection and eutaxy properties.

Perfect lattices in dimensions $n = 2$ to 7 are defined up to similarity by an integral, primitive matrix $P_n^i$, with $i = 1, \dots, 7$ if $n = 6$ and $i = 1, \dots, 33$ if $n = 7$. In dimension 8 I shall use the notation $p8dk[i]$ of my homepage, where $k \in [2, 7]$ defines the family of perfect lattices having a $k$-dimensional perfect section with the same minimum, but none of dimension $k + 1$ (if $k < 7$); we shall essentially need the case when $k = 7$, where $i$ may take 1175 values, defining lattices of decreasing density from $p8d7[1] \simeq \mathbb{E}_8$ to $p8d7[1175] \simeq \mathbb{A}_8$. $A \setminus B$

The lattice $P_6^5$ was discovered in 1957 by Barnes when he established the classification of 6-dimensional perfect lattices. He then constructed a series $P_n$ extending extending $P_6 := P_6^5$ beyond dimension 6; see [B1], [B2], and [M], Section 5.3. In this note we construct analogues of Barnes's $P_n$ relatively to $P_6^6$.

The lattice $P_6^6$ can be defined by the Gram matrix

$$P_6^6 = \begin{pmatrix} 4&1&2&2&2&2 \\ 1&4&2&2&2&2 \\ 2&2&4&1&2&2 \\ 2&2&1&4&2&2 \\ 2&2&2&2&4&1 \\ 2&2&2&2&1&4 \end{pmatrix} .$$

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Consider for $n \geq 2$ the matrix $B'_n(X)$ with entries in the polynomial ring $\mathbb{Z}[X]$, equal to 4 on the diagonal and to 2 off the diagonal, except $B_{i,i+1} = B_{i+1,i} = X$, and set $A'_n = B_n(1)$. Then $A'_6$ is the Gram matrix for $P_6^6$ displayed above. It will turn out that the matrices $A'_n$ are positive, definite, hence can be viewed as Gram matrices for lattices $L'_n$ in a convenient basis $\mathcal{B}'_n = (e_1, \dots, e_n)$. Clearly $L'_n$ is the cross-section of $L'_{n+1}$ by the hyperplane orthogonal to the vector ${e'}^*_{n+1}$ of the dual basis to $\mathcal{B}'$.

We then define $L_n$ to be $L'_n$ if $n$ is even, and the orthogonal of ${e'}^*_1 + \cdots + {e'}^*_n$ in $L_{n+1}$ if $n$ is odd.

Explicit calculations with *PARI-GP* show that we have the following isomorphisms:

$$L_5 \simeq P_5^2,\ L_6 \simeq P_6^6,\ L_7 \simeq P_7^{27},\ L'_7 \simeq P_7^{32} \text{ and } L_8 \simeq p8d7[1168]\,.$$

**Theorem 0.1.** *The lattices $L_n$, $n \geq 5$, and $L'_n$, $n \geq 6$, are extreme.*

This means that the Hermite invariant ($\gamma(L) := \frac{\min L}{\det(L)}$) attains a local maximum at $L_n$ and $L'_n$. This is equivalent to *perfection* and *eutaxy*. That the lattices $L_n$ and $L'_n$ are perfect (resp. eutactic) will be proved in Section 2 (resp. 3). Basics facts concerning perfection and eutaxy are recalled in Subsection 3.2.

## 1. Computation of determinants

**Lemma 1.1.** *The determinant of $B'_n(X)$ are as follows:*

$$(n \geq 2 \text{ even})\ \det(B'(X)) = (-1)^{n/2}\,(X-4)^{n/2}\,X^{(n-2)/2}\,(X+2n)\,;$$
$$(n \geq 3 \text{ odd})\ \det(B'(X)) = 4(-1)^{(n-1)/2}\,(X-4)^{(n-1)/2}\,X^{(n-3)/2}\,(X+n-1)\,.$$

*Proof.* Denote by $R_1, \dots, R_n$ the rows of the matrix $B'$, by $f \in \mathbb{Z}[X]$ its determinant.

We first consider even dimensions. Since $X$ occurs once in each row and column of $B(X)$, $f$ has degree $n$ and leading coefficient $a_n = \pm 1$. Developing the determinant along the first two rows shows the relation $a_n X^n = (-X^2) a_{n-2} X^{n-2}$, hence $a_n = (-1)^{n/2}$. Since the combinations $R_{i+1} - R_i$, $i \geq 1$ odd (resp. $R_{i+1} + R_i - R_1 - R_2$, $i \geq 3$ odd) are zero for $X = 4$ (resp. for $X = 0$), $f$ is divisible by $(X-4)^{n/2}$ and $X^{(n-2)/2}$, hence of the form

$$(-1)^{n/2}\,(X-4)^{n/2}\,X^{(n-2)/2}\,(X+\alpha)\,;$$

and since the components of $\sum_i R_i$ are all equal to $X + 2n$, we have $\alpha = -2n$.

Similar arguments show that for odd $n$, $f$ has leading coefficient $4(-1)^{(n-1)/2}$, and that 4, 0 are roots of $f$, of multiplicities $\frac{n-1}{2}$ and $\frac{n-3}{2}$, respectively. The combination $R_1 + \cdots + R_{n-1} - (n-1)R_n$ shows that $B'$ is also divisible by $X + (n-1)$, which proves the lemma for $B'_n$, $n$ odd. □

Lemma 1.1 shows that the $B'_n(x)$ have a positive determinant on $(0,4)$. Since they contain a sequence of principal minors of the form $B'_k$ for $k < n$, these matrices are positive definite on $(0,4)$. In particular the $A'_n$ are positive definite. This gives sense to the notation $L'_n$ of the introduction, thus also of $L_n$, which is defined as a cross-section of $L'_n$.

**Proposition 1.2.** *The determinants of the lattices of the families $L_n$ and $L'_n$ are as follows:*

(1) $\det(L_n) = \det(L'_n) = (2n+1)\,3^{n/2}$ *($n$ even).*
(2) $\det(L_n) = (n+1)\,3^{(n+1)/2}$ *($n$ odd).*
(3) $\det(L'_n) = 4n\,3^{(n-1)/2}$ *($n$ odd).*

*Proof.* Assertions (1) and (3) are direct consequences of Lemma 1.1.

To prove (2) consider the matrix $A'' = (a''_{i,j})$ with entries $4(n-1)$ on the diagonal and $-6$ off the diagonal, except $a''_{i,i+1} = a''_{i+1,i} = 2n-5$ for $i = 1, 3, \dots, 2n-1$, and the all ones vector $v$. We then check that

$$A'\,A'' = 3(2n+1)\,I_n \quad\text{and}\quad vA''\,{}^t v = \textstyle\sum_{i,j} a''_{i,j} = 3n\,,$$

which implies

$$A'^{-1} = \tfrac{1}{3(2n+1)}A'' \quad\text{and}\quad N(e_1^* + \cdots + e_n^*) = \tfrac{3n}{3(2n+1)} = \tfrac{n}{2n+1}\,.$$

This shows that the determinant of the section of $L_n$ by $(e_1^* + \cdots + e_n^*)^\perp$ is equal to $(2n+1)\,3^{n/2} \times \frac{n}{2n+1} = n\,3^{n/2}$, which gives us formula (2) after changing $n$ into $n+1$. □

## 2. Minimal vectors

Except in the statements, we assume that $E$ has even dimension $n = 2m$; the results will then be applied to the hyperplane section $L_{n-1}$ or the extension $L'_{n+1}$ of $L_n$.

We consider the three sets $o_1, o_2, o_3$ below of norm 4 vectors, which will be proved to be the orbits under $\mathrm{Aut}(L_n)$, the first two of which lie in $L_{n-1}$ :

$o_1 = \{\pm(e_j - e_i)\}$, $i < j$, $j \ne i+1$ if $i$ is odd.

$o_2 = \{\pm(e_j + e_{j+1} - e_i - e_{i+1})\}$, $i, j$ odd, $i < j$.

$o_3 = \{\pm e_i\}$.

**Proposition 2.1.** *The lattices $L_n$ have minimum 4, and for every $n \ge 4$, their sets of minimal vectors are $S = o_1 \cup o_2 \cup o_3$ if $n$ is even, and the union of the sets $o_1$ and $o_2$ relative to $L_{n+1}$ if $n$ is odd.*

*Proof.* For $x = \sum_i x_i\, e_i \in L_n$, we have

$$N(x) = 2\big(\big(\sum_i x_i\big)^2 + \sum_i x_i^2 - \sum_{i\,odd} x_i x_{i+1}\big)$$
$$= 2\big(\sum_i x_i\big)^2 + \sum_i x_i^2 + \sum_{i\,odd} x_i x_{i+1}\,.$$

Let $x \neq 0$ and denote by $k$ the number of non-zero components of $x$. We shall show that $N(x) \geq 4$ and that equality holds if and only if $x$ belongs to one of the sets $o_i$. Negating $x$ if need be, we may assume that $\sum x_i \geq 0$. We have $\sum x_i^2 \geq k$, so that we may assume that $k \leq 4$.

If $k = 1$ and $x_i \neq 0$ we have $N(x) = 4x_i^2 \geq 4$ and equality holds only if $x = e_i$.

Let now $k \geq 2$. If $|x_i| \geq 2$ for some $i$, we have $\sum x_i^2 \geq 4+(k-1) > 4$, so that we may assume that all $x_i$ are 0 or $\pm 1$.

If $k = 3$, $\sum x_i$ is odd, and we have $N(x) \geq 2+k > 4$. Hence we have $k = 2$ or 4, $\sum x_i$ is even, and if $\sum x_i \geq 2$, we have $N(x) \geq 4 + k \geq 6$, so that we may assume that $\sum x_i = 0$.

If $k = 2$, we have $x_i = \pm 1, x_j = \mp 1$ for to indices $i, j$. If $\{i, j\} = \{i, i + 1\}$, $i$ odd, we have $x_i - x_{i+1} = \pm 2$ and $N(x) = 6$. Otherwise, $x$ belongs to $o_1$.

Finally, if $k = 4$, we obviously have $N(x) \geq 4$, and equality holds if and only if all terms $x_i - x_{i+1}$ with odd $i$ are zero, that is, if and only if $x$ belongs to $o_2$. □

We state without a proof Proposition 2.2 below, which can be proved by similar arguments.

**Proposition 2.2.** *The minimal vectors of the lattices $L'_{n+1}$ which lie off $L'_n = L_n$ are $\pm e_n$ and the $\pm(e_n - e_i)$, $i = 1, \dots, n-1$.* □

**Corollary 2.3.** *The lattice $L_n$, $n \geq 5$ and $L'_n$, $n \geq 6$, are perfect.*

*Proof.* Recall (see [M], Proposition 3,5,3) that an $n$-dimensional lattice having a perfect hyperplane section with the same minimum and containing $n$ independent minimal vectors off this section is perfect. The two propositions above show that $L_n \setminus L_{n-1}$ (any $n$) and $L'_n \setminus L_{n-1}$ ($n \geq 7$ odd) contain $n$ independent minimal vectors. Since $L_5 \simeq P_5^2$ is perfect, so are all lattices $L_n$ ($n \geq 5$) and $L'_n$ ($n \geq 6$). □

**Corollary 2.4.** *The kissing numbers of the lattices $L_n$ and $L'_n$ are as follows:*

(1) $s(L_n) = s(L'_n) = \frac{n(5n-2)}{8}$ *(n even).*
(2) $s(L_n) = \frac{5(n^2-1)}{8}$ *(n odd).*
(3) $s(L'_n) = \frac{5n^2-4n+7}{8}$ *(n odd).*

*Proof.* The numbers of minimal vectors results from Propositions 2.1 and 2.2, changing $n$ into $n+1$ in case (2) and into $n-1$ in case (3). □

## 3. Automorphisms and Eutaxy

3.1. **Automorphisms.** We still assume that $E$ has even dimension $n = 2m$. We let the symmetric group $\mathfrak{S}_m$ act as a permutation group on $\{e_1, \dots, e_n\}$ by

$$\sigma\, e_{2i+1} = e_{2\sigma\, i+1} \text{ and } \sigma\, e_{2i+2} = e_{2\sigma\, i+2}\,.$$

The group $\mathfrak{S}_m$ acts as a group of automorphisms of $L_n$, and so does the larger group $G_0$ generated by $\mathfrak{S}_m$ and the $m$ transpositions $(i, i+1)$, $i$ odd. We set $G = \langle G_0, -\mathrm{Id}\rangle$. This is a semi-direct product:

$$G \simeq C_2 \times (C_2^m \cdot \mathfrak{S}_m)\,.$$

Clearly the sets $o_i$ are orbits under the action of $G$ on $S(L_n)$, and consideration of the spectrum of minimal vectors (for each $x$ minimal, the numbers of vectors having a given scalar products with $x$) easily shows that $G$ is the full automorphism group of $L_n$ and $L_{n-1}$ and that the orbits of minimal vectors are $o_1, o_2, o_3$ on $L_n$ and $o_1, o_2$ on $L_{n-1}$, except that $o_1, o_2$ collapse to a single orbit in $L_5$, which has a twice larger automorphism group.

The situation is slightly more complicated for $L'_{n+1}$. There are four orbits, namely $o'_1 := o_1$, $o'_3 := o_3$, $o'_4 = \{\pm e'_{n+1}\}$, and $o'_2$, the union of $o_2$ and the $\{\pm(e'_{n+1} - e'_i)\}$, exchanged by a twice bigger automorphism group.

3.2. **Eutaxy.** Denote by $p_F$ the orthogonal projection onto a subspace $F$ of $E$. For $x \in E$, set $p_x = p_{\mathbb{R}x}$, and for a lattice $\Lambda$, consider the set $\mathcal{E}_\Lambda := \{p_x, x \in S(\Lambda)/\{\pm\}\}$. The lattice $\Lambda$ is *perfect* if $\mathcal{E}$ is of maximal rank ($\frac{n(n+1)}{2}$) and *eutactic* if there exists a relation $\mathrm{Id} = \sum_{x\in\mathcal{E}} \lambda_x p_x$ (a *eutaxy relation*) with strictly positive coefficients $\lambda_x$.

Let $\mathcal{B}$ be a basis for $\Lambda$, let $A = \mathrm{Gram}(\mathcal{B})$, and for $x \in E$, denote by $X$ the column of components of $x \in\ \mathcal{B}$. One has $A = \mathrm{Mat}(\mathrm{Id}, \mathcal{B}, \mathcal{B}^*)$, $\mathrm{Mat}(p_x, \mathcal{B}^*, \mathcal{B}) = \frac{1}{x\cdot x}\, X^t X$, thus

$$\mathrm{Mat}(p_x, \mathcal{B}, \mathcal{B}) = \tfrac{1}{x\cdot x}\, \mathrm{Mat}_{\mathcal{B}^*}(\mathcal{B})\, X^t X = \tfrac{1}{x\cdot x}\, A X^t X\,,$$

and the eutaxy relation reads $I_n = A \sum \lambda_x X^t X$, $A^{-1} = \sum \mu_x X^t X$ with coefficients $\mu_x$ proportional to the $\lambda_x$. (Indeed, $\mu_x = \frac{\lambda_x}{\min \Lambda}$.)

An averaging argument shows that for any subgroup $G$ of $\mathrm{Aut}(\Lambda)$, the coefficients $\lambda_x$ (or $\mu_x$) may be chosen to be constant on orbits of $G$. We shall use this remark. We assume that $n$ is even and consider first $L_n$, then $L_{n-1}$, and finally $L'_{n+1}$.

Set $M_i = \sum_{x\in o_i/\pm} X^tX$. Because of the action of $G$, it suffices to consider the first three components of $M_1$, $M_2$, $M_3$ and $A'^{-1}$, namely

$$(n-2,0,-1),\ (\tfrac{n-2}{2},\tfrac{n-2}{2},-1),\ (1,0,0),\ \text{and}\ \tfrac{1}{3(2n+1)}(4(n-1),2n-5,-6)\,.$$

The eutaxy coefficients are the the solutions of the linear system

$$\begin{pmatrix} n-2 & \frac{n-2}{2} & 1 \\ 0 & \frac{n-2}{2} & 0 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{3(2n+1)} \begin{pmatrix} 4(n-1) \\ 2n-5 \\ -6 \end{pmatrix},$$

i.e., $x_1 = \frac{2(n-1)}{3(n-2)(2n+1)}$, $x_2 = \frac{2(2n-5)}{3(n-2)(2n+1)}$ and $x_3 = \frac{1}{2n+1}$.

We now consider $L_{n-1}$. Let $f = e'^*_1 + \cdots + e'^*_n$ and $H = f^\perp$, so that $L_{n-1} = L_n \cap H$. A eutaxy relation for $L_{n-1}$ may be written on the form $p_H = \sum_{S/\{\pm\}} \lambda_x p_x$, which allows to calculate in the basis we used for $L_n$. Denoting by $M$ the matrix of $p_H$ in $\mathcal{B}$ and working in $\mathcal{B}^*$ and $\mathcal{B}$, the eutaxy relation takes the form $MA'^{-1} = x_1M_1 + x_2M_2$.

We have $pH(x) = x - p_f(x) = x - \frac{c\cdot f}{f\cdot f}\,f$ and $f\cdot f = \frac{n}{2n+1}$. Setting $A'^{-1} = (\alpha_{i,j})$, we have $e^*_j = \sum_k \alpha_{k,j}e_k$ and $\sum_j \alpha_{k,j} = \frac{1}{2n+1}$, whence

$$p_H(e_i) = e_i - \tfrac{2n+1}{n}\ \textstyle\sum_j e^*_j = I_n - \tfrac{1}{n}\,J$$

where $J$ is the all ones matrix. This makes easy the calculation of $MA'^{-1}$. The result, after changing $n$ into $n+1$, is

$$(x_1,x_2) = \tfrac{2}{3(n^2-1)}\,(n-2,\tfrac{n+1}{2})\,.$$

Note that up to a scaling factor, the eutaxy coefficients are

$$(n-1,2n-5,\tfrac{3(n-2)}{2})\ (n\ \text{even})\ \text{and}\ (n-2,\tfrac{n+1}{2})\ (n\ \text{odd})\,.$$

[The scaling factor can be recovered from the orbit lengths, $(\frac{n(n-2)}{2},\frac{n(n-2)}{8},n)$ ($n$ even) and $(\frac{n^2-1}{2},\frac{n^2-1}{8})$ ($n$ odd).]

There remains to prove the eutaxy property for the series $L'_n$, $n \ge 7$ odd. We shall not give the details of the complicated calculations with the matrix $A'^{-1}$, but just state the very simple result: the eutaxy coefficients are proportional to $(4,6,8,3)$.

[The orbit lengths are $(\frac{(n-1)(n-3)}{2}, 2(n-2), \frac{(n-1)(n-3)}{8}, 1)$.]

This completes the proof of Theorem 0.1.

3.3. **Duality.** I collect here a few putative invariants of the series $L_n$ and $L'_n$ related to duality. Thus the data I list largely rely on experimentation.

Let $n \ge 6$ even. Then the first two layers of $L^*_n$ appear to be $\{\pm(e^*_1+\cdots+e^*_n)\}$ and $\{\pm e^*_1,\dots,\pm e^*_n\}$, acted on transitively by $\mathrm{Aut}(L_n)$, so that $L_{n-1}$ and $L'_{n-1}$ are the densest two cross-sections of $L_n$, isometric to $L_{n-1}$ and and $L'_{n-1}$, respectively. When $L^*_n$ is scaled to the smallest minimum which makes it integral the norms of these layers are $n$ and $\frac{4}{3}(n-4)$ if $n \equiv 4 \mod 6$, and $3n$ and $4(n-1)$ if $n \equiv 0, 2 \mod 4$, respectively.

Let now $n \geq 7$ odd. Experimentation suggests that $S(L_n^*)$ has the configuration of $S(\mathbb{A}_n^*)$, i.e., that up to sign the minimal vectors of $L_n^*$ are of the form $f_1, \dots, f_n, f_0 := f_1 + \cdots + f_n$. With the Gram matrix *P66(n)* of next Section, one could choose $f_1 = -e_1^*$ and $f_i = e_i^*$ for $i = 2, \dots, n$. However there seems to be three orbits on $S(L_n^*)$, namely $\{\pm f_1, \dots, \pm f_{n-2}\}$, $\{\pm f_{n-1}, \pm f_n\}$ and $\{\pm f_0\}$, and that orthogonality with respect to the vectors of the first orbit defines lattices isometric to $L_{n-1}$ whereas the other two orbits define non-perfect lattices.

Thus the descending chain $L_n \supset L_{n-1} \supset L_{n-2} \supset \dots$ is most certainly made of successive densest cross-sections.

As for $L'_n$, experimentation suggests that with the Gram matrix *P66a(n)* of next Section, $S({L'}_n^*)$ consists of $\pm e_n^*$ and $\pm(e_1^* + \cdots + e_n^*)$, both having an orthogonal in $L_n$ isometric to $L'_{n-1} = L_{n-1}$.

## 4. Programs

We give below *PARI-GP*-codes which produce Gram matrices for $L_n$ ($P66(n)$) and $L'_n$ ($P66a(n)$; the matrices $A'$ of Section 1). Note that $P66(n) = P66a(n)$ for even $n$). For the sake of completeness, we also give a *PARI-GP*-code for the Barnes lattices $P_n$ mentioned in the introduction.

```
{P66(n) =
local(m, a, b);
if (n == 1, return[4]); if(n%2 == 0, m = n, m = n + 1);
a = 2 * matid(m) + matrix(m, m, i, j, 2);
forstep(i = 1, m, 2, a[i, i + 1] = 1; a[i + 1, i] = 1);
  if(n%2 == 1, b = matrix(n, n, i, j, a[i + 1, j + 1]);
    for(i = 2, n - 2, b[1, i] = 1; b[i, 1] = 1);
      a = b; );
a; }

{P66a(n) =
local(m, a);
if (n == 1, return[4]); if (n%2 == 0, m = n, m = n + 1);
a = P66(m); if(n%2 == 1, a = matrix(n, n, i, j, a[i, j]));
a; }
```

$\{PBarn(n) =$
$\mathrm{local}(a);$
$\mathrm{if}(n < 4, \mathrm{print}(\text{“dimension must be at least 4”}); \mathrm{return}\,(0));$
$a = \mathrm{matrix}(n, n); \mathrm{for}(i = 2, n, a[i-1, i] = -1); \mathrm{for}(i = 3, n, a[i-2, i] = -2);$
$\mathrm{for}(i = 4, n, a[i-3, i] = 1);$
$a[1, n-1] = 1; a[2, n] = 1; a[1, n] = -2;$
$\mathrm{if}(n\%2 == 1,\ a[1, n] = -1; a[2, n] = 0; a[3, n] = 2; a[4, n] = -1;$
$a[n-2, n] = -1; a[n-1, n] = 2;\ \mathrm{for}(i = 5, n-3, a[i, n] = 0));$
$a + a\tilde{\ } + 4 * \mathrm{matid}(n); \}$

## References

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[PARI] *PARI/GP, a computer algebra system designed for fast computations in number theory*, url http://pari.math.u-bordeaux.fr/ .