EXTENSIONS OF THE PERFECT LATTICE $\mathbf{P_6^6}$

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Abstract. We construct two families L_n, L'_n of perfect lattices which extend beyond dimension 6 the six-dimensional perfect lattice P_6^6, with $L'_n \simeq L_n$ if n is even, but not if n is odd.

Introduction

We denote by E a Euclidean space, of dimension $n \geq 2$. Given a lattice $\Lambda \subset E$, its *minimum* is $\min \Lambda = \min_{x \in \Lambda \setminus \{0\}} x \cdot x$. We denote by $S(\Lambda)$ its set of *minimal vectors* (those with $x \cdot x = \min \Lambda$) and set $s(\Lambda) = \frac{1}{2}\,|S(\Lambda|$.

We refer to [M], in particular to chapters 1 and 3 for general definitions relative to (Euclidean) lattices, including the perfection and eutaxy properties.

Perfect lattices in dimensions $n = 2$ to 7 are defined up to similarity by an integral, primitive matrix P_n^i, with $i = 1, \dots, 7$ if $n = 6$ and $i = 1, \dots, 33$ if $n = 7$. In dimension 8 I shall use the notation $p8dk[i]$ of my homepage, where $k \in [2, 7]$ defines the family of perfect lattices having a k-dimensional perfect section with the same minimum, but none of dimension $k + 1$ (if $k < 7$); we shall essentially need the case when $k = 7$, where i may take 1175 values, defining lattices of decreasing density from $p8d7[1] \simeq \mathbb{E}_8$ to $p8d7[1175] \simeq \mathbb{A}_8$. $A \setminus B$

The lattice P_6^5 was discovered in 1957 by Barnes when he established the classification of 6-dimensional perfect lattices. He then constructed a series P_n extending extending $P_6 := P_6^5$ beyond dimension 6; see [B1], [B2], and [M], Section 5.3. In this note we construct analogues of Barnes's P_n relatively to P_6^6.

The lattice P_6^6 can be defined by the Gram matrix

$$P_6^6 = \begin{pmatrix} 4&1&2&2&2&2\\ 1&4&2&2&2&2\\ 2&2&4&1&2&2\\ 2&2&1&4&2&2\\ 2&2&2&2&4&1\\ 2&2&2&2&1&4 \end{pmatrix}.$$

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Consider for $n \geq 2$ the matrix $B'_n(X)$ with entries in the polynomial ring $\mathbb{Z}[X]$, equal to 4 on the diagonal and to 2 off the diagonal, except $B_{i,i+1} = B_{i+1,i} = X$, and set $A'_n = B_n(1)$. Then A'_6 is the Gram matrix for P_6^6 displayed above. It will turn out that the matrices A'_n are positive, definite, hence can be viewed as Gram matrices for lattices L'_n in a convenient basis $\mathcal{B}'_n = (e_1, \dots, e_n)$. Clearly L'_n is the cross-section of L'_{n+1} by the hyperplane orthogonal to the vector ${e'}^*_{n+1}$ of the dual basis to $\mathcal{B}'$.

We then define L_n to be L'_n if n is even, and the orthogonal of ${e'}^*_1 + \cdots + {e'}^*_n$ in L_{n+1} if n is odd.

Explicit calculations with *PARI-GP* show that we have the following isomorphisms:

$$L_5 \simeq P_5^2,\ L_6 \simeq P_6^6,\ L_7 \simeq P_7^{27},\ L'_7 \simeq P_7^{32} \text{ and } L_8 \simeq p8d7[1168]\,.$$

Theorem 0.1. *The lattices L_n, $n \geq 5$, and L'_n, $n \geq 6$, are extreme.*

This means that the Hermite invariant ($\gamma(L) := \frac{\min L}{\det(L)}$) attains a local maximum at L_n and L'_n. This is equivalent to *perfection* and *eutaxy*. That the lattices L_n and L'_n are perfect (resp. eutactic) will be proved in Section 2 (resp. 3). Basics facts concerning perfection and eutaxy are recalled in Subsection 3.2.

1. Computation of determinants

Lemma 1.1. *The determinant of $B'_n(X)$ are as follows:*

$$(n \geq 2 \textit{ even})\ \det(B'(X)) = (-1)^{n/2}\,(X-4)^{n/2}\,X^{(n-2)/2}\,(X+2n)\,;$$
$$(n \geq 3 \textit{ odd})\ \det(B'(X)) = 4(-1)^{(n-1)/2}\,(X-4)^{(n-1)/2}\,X^{(n-3)/2}\,(X+n-1)\,.$$

Proof. Denote by $R_1, \dots, R_n$ the rows of the matrix B', by $f \in \mathbb{Z}[X]$ its determinant.

We first consider even dimensions. Since X occurs once in each row and column of $B(X)$, f has degree n and leading coefficient $a_n = \pm 1$. Developing the determinant along the first two rows shows the relation $a_n X^n = (-X^2) a_{n-2} X^{n-2}$, hence $a_n = (-1)^{n/2}$. Since the combinations $R_{i+1} - R_i$, $i \geq 1$ odd (resp. $R_{i+1} + R_i - R_1 - R_2$, $i \geq 3$ odd) are zero for $X = 4$ (resp. for $X = 0$), f is divisible by $(X-4)^{n/2}$ and $X^{(n-2)/2}$, hence of the form

$$(-1)^{n/2}\,(X-4)^{n/2}\,X^{(n-2)/2}\,(X+\alpha)\,;$$

and since the components of $\sum_i R_i$ are all equal to $X + 2n$, we have $\alpha = -2n$.

Similar arguments show that for odd n, f has leading coefficient $4(-1)^{(n-1)/2}$, and that 4, 0 are roots of f, of multiplicities $\frac{n-1}{2}$ and $\frac{n-3}{2}$, respectively. The combination $R_1 + \cdots + R_{n-1} - (n-1)R_n$ shows that B' is also divisible by $X + (n-1)$, which proves the lemma for B'_n, n odd. □

Lemma 1.1 shows that the $B'_n(x)$ have a positive determinant on $(0,4)$. Since they contain a sequence of principal minors of the form B'_k for $k < n$, these matrices are positive definite on $(0,4)$. In particular the A'_n are positive definite. This gives sense to the notation L'_n of the introduction, thus also of L_n, which is defined as a cross-section of L'_n.

Proposition 1.2. *The determinants of the lattices of the families L_n and L'_n are as follows:*

(1) $\det(L_n) = \det(L'_n) = (2n+1)\,3^{n/2}$ *(n even).*
(2) $\det(L_n) = (n+1)\,3^{(n+1)/2}$ *(n odd).*
(3) $\det(L'_n) = 4n\,3^{(n-1)/2}$ *(n odd).*

Proof. Assertions (1) and (3) are direct consequences of Lemma 1.1.

To prove (2) consider the matrix $A'' = (a''_{i,j})$ with entries $4(n-1)$ on the diagonal and -6 off the diagonal, except $a''_{i,i+1} = a''_{i+1,i} = 2n-5$ for $i = 1, 3, \dots, 2n-1$, and the all ones vector v. We then check that

$$A'\,A'' = 3(2n+1)\,I_n \quad \text{and} \quad vA''\,{}^t v = \textstyle\sum_{i,j} a''_{i,j} = 3n\,,$$

which implies

$$A'^{-1} = \tfrac{1}{3(2n+1)}A'' \quad \text{and} \quad N(e_1^* + \cdots + e_n^*) = \tfrac{3n}{3(2n+1)} = \tfrac{n}{2n+1}\,.$$

This shows that the determinant of the section of L_n by $(e_1^* + \cdots + e_n^*)^\perp$ is equal to $(2n+1)\,3^{n/2} \times \frac{n}{2n+1} = n\,3^{n/2}$, which gives us formula (2) after changing n into $n+1$. □

2. Minimal vectors

Except in the statements, we assume that E has even dimension $n = 2m$; the results will then be applied to the hyperplane section L_{n-1} or the extension L'_{n+1} of L_n.

We consider the three sets o_1, o_2, o_3 below of norm 4 vectors, which will be proved to be the orbits under $\mathrm{Aut}(L_n)$, the first two of which lie in L_{n-1} :

$o_1 = \{\pm(e_j - e_i)\}$, $i < j$, $j \neq i+1$ if i is odd.

$o_2 = \{\pm(e_j + e_{j+1} - e_i - e_{i+1})\}$, i, j odd, $i < j$.

$o_3 = \{\pm e_i\}$.

Proposition 2.1. *The lattices L_n have minimum 4, and for every $n \geq 4$, their sets of minimal vectors are $S = o_1 \cup o_2 \cup o_3$ if n is even, and the union of the sets o_1 and o_2 relative to L_{n+1} if n is odd.*

Proof. For $x = \sum_i x_i\, e_i \in L_n$, we have

$$\begin{aligned} N(x) &= 2\big(\big(\sum_i x_i\big)^2 + \sum_i x_i^2 - \sum_{i\,odd} x_i x_{i+1}\big) \\ &= 2\big(\sum_i x_i\big)^2 + \sum_i x_i^2 + \sum_{i\,odd} x_i x_{i+1}\,. \end{aligned}$$

Let $x \neq 0$ and denote by k the number of non-zero components of x. We shall show that $N(x) \geq 4$ and that equality holds if and only if x belongs to one of the sets o_i. Negating x if need be, we may assume that $\sum x_i \geq 0$. We have $\sum x_i^2 \geq k$, so that we may assume that $k \leq 4$.

If $k = 1$ and $x_i \neq 0$ we have $N(x) = 4x_i^2 \geq 4$ and equality holds only if $x = e_i$.

Let now $k \geq 2$. If $|x_i| \geq 2$ for some i, we have $\sum x_i^2 \geq 4+(k-1) > 4$, so that we may assume that all x_i are 0 or ± 1.

If $k = 3$, $\sum x_i$ is odd, and we have $N(x) \geq 2+k > 4$. Hence we have $k = 2$ or 4, $\sum x_i$ is even, and if $\sum x_i \geq 2$, we have $N(x) \geq 4 + k \geq 6$, so that we may assume that $\sum x_i = 0$.

If $k = 2$, we have $x_i = \pm 1, x_j = \mp 1$ for to indices i, j. If $\{i, j\} = \{i, i+1\}$, i odd, we have $x_i - x_{i+1} = \pm 2$ and $N(x) = 6$. Otherwise, x belongs to o_1.

Finally, if $k = 4$, we obviously have $N(x) \geq 4$, and equality holds if and only if all terms $x_i - x_{i+1}$ with odd i are zero, that is, if and only if x belongs to o_2. □

We state without a proof Proposition 2.2 below, which can be proved by similar arguments.

Proposition 2.2. *The minimal vectors of the lattices L'_{n+1} which lie off $L'_n = L_n$ are $\pm e_n$ and the $\pm(e_n - e_i)$, $i = 1, \dots, n-1$.* □

Corollary 2.3. *The lattice L_n, $n \geq 5$ and L'_n, $n \geq 6$, are perfect.*

Proof. Recall (see [M], Proposition 3,5,3) that an n-dimensional lattice having a perfect hyperplane section with the same minimum and containing n independent minimal vectors off this section is perfect. The two propositions above show that $L_n \setminus L_{n-1}$ (any n) and $L'_n \setminus L_{n-1}$ ($n \geq 7$ odd) contain n independent minimal vectors. Since $L_5 \simeq P_5^2$ is perfect, so are all lattices L_n ($n \geq 5$) and L'_n ($n \geq 6$). □

Corollary 2.4. *The kissing numbers of the lattices L_n and L'_n are as follows:*

(1) $s(L_n) = s(L'_n) = \frac{n(5n-2)}{8}$ *(n even).*
(2) $s(L_n) = \frac{5(n^2-1)}{8}$ *(n odd).*
(3) $s(L'_n) = \frac{5n^2-4n+7}{8}$ *(n odd).*

Proof. The numbers of minimal vectors results from Propositions 2.1 and 2.2, changing n into $n+1$ in case (2) and into $n-1$ in case (3). $\square$

3. Automorphisms and Eutaxy

3.1. **Automorphisms.** We still assume that E has even dimension $n = 2m$. We let the symmetric group $\mathfrak{S}_m$ act as a permutation group on $\{e_1, \dots, e_n\}$ by

$$\sigma\, e_{2i+1} = e_{2\sigma\, i+1} \text{ and } \sigma\, e_{2i+2} = e_{2\sigma\, i+2}\,.$$

The group $\mathfrak{S}_m$ acts as a group of automorphisms of L_n, and so does the larger group G_0 generated by $\mathfrak{S}_m$ and the m transpositions $(i, i+1)$, i odd. We set $G = \langle G_0, -\mathrm{Id}\rangle$. This is a semi-direct product:

$$G \simeq C_2 \times (C_2^m \cdot \mathfrak{S}_m)\,.$$

Clearly the sets o_i are orbits under the action of G on $S(L_n)$, and consideration of the spectrum of minimal vectors (for each x minimal, the numbers of vectors having a given scalar products with x) easily shows that G is the full automorphism group of L_n and L_{n-1} and that the orbits of minimal vectors are o_1, o_2, o_3 on L_n and o_1, o_2 on L_{n-1}, except that o_1, o_2 collapse to a single orbit in L_5, which has a twice larger automorphism group.

The situation is slightly more complicated for L'_{n+1}. There are four orbits, namely $o'_1 := o_1$, $o'_3 := o_3$, $o'_4 = \{\pm e'_{n+1}\}$, and o'_2, the union of o_2 and the $\{\pm(e'_{n+1} - e'_i)\}$, exchanged by a twice bigger automorphism group.

3.2. **Eutaxy.** Denote by p_F the orthogonal projection onto a subspace F of E. For $x \in E$, set $p_x = p_{\mathbb{R}x}$, and for a lattice Λ, consider the set $\mathcal{E}_\Lambda := \{p_x, x \in S(\Lambda)/\{\pm\}\}$. The lattice Λ is *perfect* if $\mathcal{E}$ is of maximal rank ($\frac{n(n+1)}{2}$) and *eutactic* if there exists a relation $\mathrm{Id} = \sum_{x\in\mathcal{E}} \lambda_x p_x$ (a *eutaxy relation*) with strictly positive coefficients λ_x.

Let $\mathcal{B}$ be a basis for Λ, let $A = \mathrm{Gram}(\mathcal{B})$, and for $x \in E$, denote by X the column of components of $x \in\ \mathcal{B}$. One has $A = \mathrm{Mat}(\mathrm{Id}, \mathcal{B}, \mathcal{B}^*)$, $\mathrm{Mat}(p_x, \mathcal{B}^*, \mathcal{B}) = \frac{1}{x\cdot x}\, X^t X$, thus

$$\mathrm{Mat}(p_x, \mathcal{B}, \mathcal{B}) = \tfrac{1}{x\cdot x}\, \mathrm{Mat}_{\mathcal{B}^*}(\mathcal{B})\, X^t X = \tfrac{1}{x\cdot x}\, A X^t X\,,$$

and the eutaxy relation reads $I_n = A \sum \lambda_x X^t X$, $A^{-1} = \sum \mu_x X^t X$ with coefficients μ_x proportional to the λ_x. (Indeed, $\mu_x = \frac{\lambda_x}{\min \Lambda}$.)

An averaging argument shows that for any subgroup G of $\mathrm{Aut}(\Lambda)$, the coefficients λ_x (or μ_x) may be chosen to be constant on orbits of G. We shall use this remark. We assume that n is even and consider first L_n, then L_{n-1}, and finally L'_{n+1}.

Set $M_i = \sum_{x\in o_i/\pm} X^tX$. Because of the action of G, it suffices to consider the first three components of M_1, M_2, M_3 and A'^{-1}, namely

$(n-2,0,-1)$, $(\frac{n-2}{2},\frac{n-2}{2},-1)$, $(1,0,0)$, and $\frac{1}{3(2n+1)}(4(n-1),2n-5,-6)$.

The eutaxy coefficients are the the solutions of the linear system

$$\begin{pmatrix} n-2 & \frac{n-2}{2} & 1 \\ 0 & \frac{n-2}{2} & 0 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{3(2n+1)}\begin{pmatrix} 4(n-1) \\ 2n-5 \\ -6 \end{pmatrix},$$

i.e., $x_1 = \frac{2(n-1)}{3(n-2)(2n+1)}$, $x_2 = \frac{2(2n-5)}{3(n-2)(2n+1)}$ and $x_3 = \frac{1}{2n+1}$.

We now consider L_{n-1}. Let $f = {e'}_1^* + \cdots + {e'}_n^*$ and $H = f^\perp$, so that $L_{n-1} = L_n \cap H$. A eutaxy relation for L_{n-1} may be written on the form $p_H = \sum_{S/\{\pm\}} \lambda_x p_x$, which allows to calculate in the basis we used for L_n. Denoting by M the matrix of p_H in $\mathcal{B}$ and working in $\mathcal{B}^*$ and $\mathcal{B}$, the eutaxy relation takes the form $MA'^{-1} = x_1M_1 + x_2M_2$.

We have $pH(x) = x - p_f(x) = x - \frac{c\cdot f}{f\cdot f}\,f$ and $f\cdot f = \frac{n}{2n+1}$. Setting $A'^{-1} = (\alpha_{i,j})$, we have $e_j^* = \sum_k \alpha_{k,j}e_k$ and $\sum_j \alpha_{k,j} = \frac{1}{2n+1}$, whence

$$p_H(e_i) = e_i - \frac{2n+1}{n}\,\sum_j e_j^* = I_n - \frac{1}{n}\,J$$

where J is the all ones matrix. This makes easy the calculation of MA'^{-1}. The result, after changing n into $n+1$, is

$$(x_1,x_2) = \frac{2}{3(n^2-1)}\,(n-2,\frac{n+1}{2}).$$

Note that up to a scaling factor, the eutaxy coefficients are

$$(n-1,2n-5,\frac{3(n-2)}{2})\ (n\text{ even}) \text{ and } (n-2,\frac{n+1}{2})\ (n\text{ odd}).$$

[The scaling factor can be recovered from the orbit lengths, $(\frac{n(n-2)}{2},\frac{n(n-2)}{8},n)$ (n even) and $(\frac{n^2-1}{2},\frac{n^2-1}{8})$ (n odd).]

There remains to prove the eutaxy property for the series L'_n, $n\ge 7$ odd. We shall not give the details of the complicated calculations with the matrix A'^{-1}, but just state the very simple result: the eutaxy coefficients are proportional to $(4,6,8,3)$.

[The orbit lengths are $(\frac{(n-1)(n-3)}{2},2(n-2),\frac{(n-1)(n-3)}{8},1)$.]

This completes the proof of Theorem 0.1.

3.3. **Duality.** I collect here a few putative invariants of the series L_n and L'_n related to duality. Thus the data I list largely rely on experimentation.

Let $n\ge 6$ even. Then the first two layers of L_n^* appear to be $\{\pm(e_1^*+\cdots+e_n^*)\}$ and $\{\pm e_1^*,\dots,\pm e_n^*\}$, acted on transitively by $\mathrm{Aut}(L_n)$, so that L_{n-1} and L'_{n-1} are the densest two cross-sections of L_n, isometric to L_{n-1} and and L'_{n-1}, respectively. When L_n^* is scaled to the smallest minimum which makes it integral the norms of these layers are n and $\frac{4}{3}(n-4)$ if $n \equiv 4 \mod 6$, and $3n$ and $4(n-1)$ if $n \equiv 0,2 \mod 4$, respectively.

Let now $n \geq 7$ odd. Experimentation suggests that $S(L_n^*)$ has the configuration of $S(\mathbb{A}_n^*)$, i.e., that up to sign the minimal vectors of L_n^* are of the form $f_1, \dots, f_n, f_0 := f_1 + \dots + f_n$. With the Gram matrix *P66(n)* of next Section, one could choose $f_1 = -e_1^*$ and $f_i = e_i^*$ for $i = 2, \dots, n$. However there seems to be three orbits on $S(L_n^*)$, namely $\{\pm f_1, \dots, \pm f_{n-2}\}$, $\{\pm f_{n-1}, \pm f_n\}$ and $\{\pm f_0\}$, and that orthogonality with respect to the vectors of the first orbit defines lattices isometric to L_{n-1} whereas the other two orbits define non-perfect lattices.

Thus the descending chain $L_n \supset L_{n-1} \supset L_{n-2} \supset \dots$ is most certainly made of successive densest cross-sections.

As for L'_n, experimentation suggests that with the Gram matrix *P66a(n)* of next Section, $S({L'}_n^*)$ consists of $\pm e_n^*$ and $\pm(e_1^* + \dots + e_n^*)$, both having an orthogonal in L_n isometric to $L'_{n-1} = L_{n-1}$.

4. Programs

We give below *PARI-GP*-codes which produce Gram matrices for L_n ($P66(n)$) and L'_n ($P66a(n)$; the matrices A' of Section 1). Note that $P66(n) = P66a(n)$ for even n). For the sake of completeness, we also give a *PARI-GP*-code for the Barnes lattices P_n mentioned in the introduction.

```
{P66(n) =
local(m, a, b);
if (n == 1, return[4]); if(n%2 == 0, m = n, m = n + 1);
a = 2 * matid(m) + matrix(m, m, i, j, 2);
forstep(i = 1, m, 2, a[i, i + 1] = 1; a[i + 1, i] = 1);
  if(n%2 == 1, b = matrix(n, n, i, j, a[i + 1, j + 1]);
    for(i = 2, n − 2, b[1, i] = 1; b[i, 1] = 1);
      a = b; );
a; }
```

```
{P66a(n) =
local(m, a);
if (n == 1, return[4]); if (n%2 == 0, m = n, m = n + 1);
a = P66(m); if(n%2 == 1, a = matrix(n, n, i, j, a[i, j]));
a; }
```

$\{PBarn(n) =$
$\mathrm{local}(a);$
$\mathrm{if}(n < 4, \mathrm{print}(\text{“dimension must be at least 4”}); \mathrm{return}\,(0));$
$a = \mathrm{matrix}(n, n); \mathrm{for}(i = 2, n, a[i-1, i] = -1); \mathrm{for}(i = 3, n, a[i-2, i] = -2);$
$\mathrm{for}(i = 4, n, a[i-3, i] = 1);$
$a[1, n-1] = 1; a[2, n] = 1; a[1, n] = -2;$
$\mathrm{if}(n\%2 == 1,\ a[1, n] = -1; a[2, n] = 0; a[3, n] = 2; a[4, n] = -1;$
$a[n-2, n] = -1; a[n-1, n] = 2;\ \mathrm{for}(i = 5, n-3, a[i, n] = 0));$
$a + a\text{\textasciitilde}{+}4 * \mathrm{matid}(n); \}$

References

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[PARI] *PARI/GP, a computer algebra system designed for fast computations in number theory*, url http://pari.math.u-bordeaux.fr/ .