VORONOI COMPLEXES IN HIGHER DIMENSIONS, COHOMOLOGY OF $GL_N(\mathbb{Z})$ FOR $N \geqslant 8$ AND THE TRIVIALITY OF $K_8(\mathbb{Z})$

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Abstract. We enumerate the low dimensional cells in the Voronoi cell complexes attached to the modular groups $SL_N(\mathbb{Z})$ and $GL_N(\mathbb{Z})$ for $N = 8, 9, 10, 11$, using quotient sublattice techniques for $N = 8, 9$ and linear programming methods for higher dimensions. These enumerations allow us to compute some cohomology of these groups and prove that $K_8(\mathbb{Z}) = 0$. We deduce from it new knowledge on the Kummer-Vandiver conjecture.

Let $N \geqslant 1$ be an integer and let $SL_N(\mathbb{Z})$ be the modular group of integral matrices with determinant one. Our goal is to compute its cohomology groups with trivial coefficients, i.e. $H^q(SL_N(\mathbb{Z}); \mathbb{Z})$. These are known in the cases $N \leqslant 7$: $N = 2$ is classical (e.g. II.7, Ex.3 of [8]), $N = 3$ is due to Soulé [47], $N = 4$ is due to Lee and Szczarba [38], and $N = 5, 6, 7$ are due to Elbaz-Vincent, Gangl and Soulé [19].

In Theorem 5.2 below, we give partial information for the cases $N = 8, 9, 10$. For these calculations we follow mainly the methods of [38] and [19], by investigating the Voronoi complexes associated to these modular groups.

Recall that a *perfect form* in N variables is a positive definite real quadratic form h on $\mathbb{R}^N$ with given minimum m which is uniquely determined by its set of integral minimal vectors [40](§3.2). Voronoi proved in [52] that there are finitely many perfect forms of rank N, modulo the action of $SL_N(\mathbb{Z})$. These are known for $N \leqslant 8$ (see §1 below). These finitely many orbits of perfect forms give the top-dimensional generators in the Voronoi complex; the rest of the complex is constructed from these perfect forms. Unfortunately, we cannot work with the full Voronoi complex for $N = 8$ due to its size, which is beyond our computing capabilities, and we do not have complete information for $N > 8$. However, it turns out that it *is* possible to obtain partial information on the top and bottom parts of the Voronoi complexes for $8 \leqslant N \leqslant 10$ and conjecturally $N = 11$. In particular, we can enumerate the cells of lowest dimension explicitly using methods based on sublattices and relative index (cf. §2). For other cases, we can use linear programming in order to full enumeration in given cellular dimensions (cf. §3).

Voronoi used perfect forms to define a cell decomposition of the space X_N^* of positive real quadratic forms, whose kernel is defined over $\mathbb{Q}$. This cell decomposition (cf. §1.2) is invariant under $SL_N(\mathbb{Z})$, hence it can be used to give a chain complex which computes the equivariant homology of X_N^* modulo its boundary: this is the Voronoi complex. On the other hand, this equivariant homology turns out to be isomorphic to the groups $H_q(SL_N(\mathbb{Z}); \mathrm{St}_N)$, where St_N is the Steinberg

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module (see [6] and §1.2.3 below). Finally, Borel–Serre duality asserts that the homology $H_*(SL_N(\mathbb{Z}); \mathrm{St}_N)$ is dual to the cohomology $H^*(SL_N(\mathbb{Z}); \mathbb{Z})$ (modulo torsion at primes $\leqslant N+1$). Thus the results mentioned above give partial information about the cohomology of modular groups.

We will use this to obtain information about the algebraic K-theory of the integers. In particular, we will prove that the group $K_8(\mathbb{Z})$ is trivial (Theorem 6.4) and discuss its consequences for the Kummer–Vandiver conjecture (cf. §7).

Organization of paper: In §1, we recall the Voronoi theory of perfect forms and the Voronoi complex which computes the homology groups $H_q(\Gamma; \mathrm{St}_\Gamma)$ with $\Gamma = SL_N(\mathbb{Z})$ or $GL_N(\mathbb{Z})$. In §2, we give an explicit enumeration of the low dimensional cells of the Voronoi complexes associated to Γ. In §3, we present another method based on linear programming. In §4 we give a partial description of the Voronoi complex associated to modular groups of rank $N = 8$ up to 12. In §5 we compute some homology groups of Γ with coefficients in the Steinberg module and we explain how to compute part of the cohomology of $SL_N(\mathbb{Z})$ and $GL_N(\mathbb{Z})$ (modulo torsion) for $N \geqslant 8$. In §6, we use these results to get some information on $K_m(\mathbb{Z})$ for $m \geqslant 8$ and in particular show that $K_8(\mathbb{Z})$ is trivial. In §7 we give some arithmetic applications.

Some of the enumerations of configurations of vectors had already been announced and used in [26]. Results concerning the triviality of $K_8(\mathbb{Z})$ had already been announced in [18, 33].

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Contents

1. The Voronoi reduction theory

In this Section we recall some aspects of the Voronoi reduction theory [52, 40].

1.1. **Perfect forms.** Let $N \geqslant 2$ be an integer. We let C_N be the set of positive definite real quadratic forms in N variables (i.e., real quadratic forms q such that $q(v) > 0$ for any non-zero $v \in \mathbb{R}^N$). Given $h \in C_N$, let $m(h)$ be the finite set of minimal vectors of h, i.e. vectors $v \in \mathbb{Z}^N$, $v \neq 0$, such that $h(v)$ is minimal. A form h is called *perfect* when $m(h)$ determines h up to scalar: if $h' \in C_N$ is such that $m(h') = m(h)$, then h' is proportional to h.

***Example* 1.1.** The form $h(x, y) = x^2 + y^2$ has minimum 1 and precisely 4 minimal vectors $\pm(1, 0)$ and $\pm(0, 1)$. This form is not perfect, because there is an infinite number of positive definite quadratic forms having these minimal vectors, namely the forms $h(x, y) = x^2 + axy + y^2$ where a is a non-negative real number less than 1. By contrast, the form $h(x, y) = x^2 + xy + y^2$ has also minimum 1 and has exactly 6 minimal vectors, viz. the ones above and $\pm(1, -1)$. Hence this form is perfect.

Denote by C_N^* the set of non-negative real quadratic forms on $\mathbb{R}^N$ (i.e., real quadratic forms q such that $q(v) \geqslant 0$ for any $v \in \mathbb{R}^N$) the kernel of which is spanned by a proper linear subspace of $\mathbb{Q}^N$, by X_N^* the quotient of C_N^* by positive real homotheties, and by $\pi : C_N^* \to X_N^*$ the projection. Let $X_N = \pi(C_N)$ and $\partial X_N^* = X_N^* - X_N$. Let Γ be either $GL_N(\mathbb{Z})$ or $SL_N(\mathbb{Z})$. The group Γ acts on C_N^* and X_N^* on the right by the formula

$$h \cdot \gamma = \gamma^t \, h \, \gamma \,, \quad \gamma \in \Gamma \,, \; h \in C_N^* \,,$$

where h is viewed as a symmetric matrix and γ^t is the transpose of the matrix γ. Voronoi proved that there are only finitely many perfect forms modulo the action of Γ and multiplication by positive real numbers ([52], Thm. p.110).
Table 1 gives the current state of the art on the enumeration of perfect forms.

rank	1	2	3	4	5	6	7	8	9
# classes	1	1	1	2	3	7	33	10916	2237251040

TABLE 1. Known results on the number of perfect forms up to dimension 9

The classification of perfect forms of rank 8 was achieved by Dutour Sikirić, Schürmann and Vallentin [15, 46], and for dimension 9 has been done by Dutour Sikirić and van Woerden [16]. The corresponding classification for rank 7 was completed by Jaquet [29], for rank 6 by Barnes [3], for rank 5 and 4 by Korkine and Zolotarev [32, 31], for dimension 3 by Gauss [23] and for dimension 2 by Lagrange [36]. We refer to the book of Martinet [40] for more details on the results up to rank 7. While the classification of perfect forms of higher rank is not well understood, we know from Bacher [2] that the number of representatives grows at least exponentially with the rank and from van Woerden [51] is bounded by $e^{O(d^2 \log(d))}$ for perfect forms of rank d.

1.2. **The Voronoi complex.**

***Notation* 1.2.** For any positive integer n we let $\mathcal{S}_n$ be the class of finite abelian groups the order of which has only prime factors less than or equal to n.

1.2.1. *The cell complex.* Given $v \in \mathbb{Z}^N - \{0\}$ we let $\hat{v} \in C_N^*$ be the form defined by

$$\hat{v}(x) = (v \mid x)^2\,,\ x \in \mathbb{R}^N\,,$$

where $(v \mid x)$ is the scalar product of v and x. The *convex hull in* X_N^* of a finite subset $B \subset \mathbb{Z}^N - \{\mathbf{0}\}$ is the subset of X_N^* which is the image under π of the quadratic forms $\sum_j \lambda_j \hat{v}_j \in C_N^*$, where $v_j \in B$ and $\lambda_j \geqslant 0$. For any perfect form h, we let $\sigma(h) \subset X_N^*$ be the convex hull of the set $m(h)$ of its minimal vectors. Voronoi proved in [52], §§8-15, that the cells $\sigma(h)$ and their intersections, as h runs over all perfect forms, define a cell decomposition of X_N^*, which is invariant under the action of Γ. We endow X_N^* with the corresponding CW-topology. If τ is a closed cell in X_N^* and h a perfect form with $\tau \subset \sigma(h)$, we let $m(\tau)$ be the set of vectors v in $m(h)$ such that $\hat{v}$ lies in τ. Any closed cell τ is the convex hull of $m(\tau)$, and for any two closed cells τ, τ' in X_N^* we have $m(\tau) \cap m(\tau') = m(\tau \cap \tau')$.

We shall now recall an explicit description of the Voronoi complex and its differential from [19].

Let $d(N) = N(N+1)/2 - 1$ be the dimension of X_N^* and $n \leqslant d(N)$ a natural integer. We denote by $\Sigma_n^\star = \Sigma_n^\star(\Gamma)$ a set of representatives, modulo the action of Γ, of those cells of dimension n in X_N^* which meet X_N, and by $\Sigma_n = \Sigma_n(\Gamma) \subset \Sigma_n^\star(\Gamma)$ the cells σ for which any element of the stabilizer Γ_σ of σ in Γ preserves orientation.

Let V_n be the free abelian group generated by Σ_n. We define as follows a map

$$d_n \colon V_n \to V_{n-1}\,.$$

For each closed cell σ in X_N^* we fix an orientation of σ, i.e. an orientation of the real vector space $\mathbb{R}(\sigma)$ of symmetric matrices spanned by the forms $\hat{v}$ with $v \in m(\sigma)$. Let $\sigma \in \Sigma_n$ and let τ' be a face of σ which is equivalent under Γ to an element in Σ_{n-1} (i.e. τ' neither lies on the boundary nor has elements in its stabilizer reversing the orientation). Given a positive basis B' of $\mathbb{R}(\tau')$ we get a basis B of $\mathbb{R}(\sigma) \supset \mathbb{R}(\tau')$ by appending to B' a vector $\hat{v}$, where $v \in m(\sigma) - m(\tau')$. We let $\varepsilon(\tau', \sigma) = \pm 1$ be the sign of the orientation of B in the oriented vector space $\mathbb{R}(\sigma)$ (this sign does not depend on the choice of v).

Next, let $\tau \in \Sigma_{n-1}$ be the (unique) cell equivalent to τ' and let $\gamma \in \Gamma$ be such that $\tau' = \tau \cdot \gamma$. We define $\eta(\tau, \tau') = 1$ (resp. $\eta(\tau, \tau') = -1$) when γ is compatible (resp. incompatible) with the chosen orientations of $\mathbb{R}(\tau)$ and $\mathbb{R}(\tau')$.

Finally, if $\sigma \in \Sigma_n$ and $\tau \in \Sigma_{n-1}$, we define the incidence number $[\sigma : \tau]$ for the Voronoi complex as

$$[\sigma : \tau] = \sum_{\tau'} \eta(\tau, \tau')\, \varepsilon(\tau', \sigma)\,, \tag{1}$$

where τ' runs through the set of faces of σ which are equivalent to τ. If τ is not equivalent to a face of σ, we set $[\sigma : \tau] = 0$. The following map is thus well defined

$$d_n(\sigma) = \sum_{\tau \in \Sigma_{n-1}} [\sigma : \tau]\, \tau\,. \tag{2}$$

It turns out that the map d generalizes the usual differential of regular CW-complex to the case of the Voronoi complex (which is not regular CW-complex).

1.2.2. *The associated equivariant spectral sequence.* According to Section VII.7 of [8], there is a spectral sequence E^r_{pq} converging to the equivariant homology groups $H^\Gamma_{p+q}(X^*_N, \partial X^*_N; \mathbb{Z})$ of the pair $(X^*_N, \partial X^*_N)$, with E^1-page given by

$$E^1_{pq} = \bigoplus_{\sigma\in\Sigma^\star_p} H_q(\Gamma_\sigma; \mathbb{Z}_\sigma)\,,$$

where $\mathbb{Z}_\sigma$ is the orientation module of the cell σ and, as above, $\Sigma^\star_p$ is a set of representatives, modulo Γ, of the p-cells σ in X^*_N which meet X_N. Notice that the action of Γ_σ on $\mathbb{Z}_\sigma$ is given by η described above. Since σ meets X_N, its stabilizer Γ_σ is finite and, by Lemma 4.1 in §3 below, the order of Γ_σ is divisible only by primes $p \leqslant N+1$. Therefore, when q is positive, the group $H_q(\Gamma_\sigma; \mathbb{Z}_\sigma)$ lies in $\mathcal{S}_{N+1}$.

When Γ_σ happens to contain an element which changes the orientation of σ, the group $H_0(\Gamma_\sigma; \mathbb{Z}_\sigma)$ is killed by 2, otherwise $H_0(\Gamma_\sigma; \mathbb{Z}_\sigma) \cong \mathbb{Z}$. Therefore, modulo $\mathcal{S}_2$, we have

$$E^1_{n\,0} = \bigoplus_{\sigma\in\Sigma_n} \mathbb{Z}_\sigma\,,$$

and the choice of an orientation for each cell σ gives an isomorphism between $E^1_{n\,0}$ and V_n.

Proposition 1.3. [19, §3.3, p.591-592] *The differential*

$$d^1_n\colon E^1_{n\,0} \to E^1_{n-1,0}$$

coincides, up to sign, with the map d_n defined in 1.2.1.

As pointed out on p.589 of [19], the identity $d_{n-1}\circ d_n = 0$ gives us a non-trivial test of our explicit computations.

***Notation* 1.4.** The resulting complex $(V_\bullet, d_\bullet)$ is denoted by Vor_Γ, and is called the *Voronoi complex.*

1.2.3. *The Steinberg module.* Let T_N be the spherical Tits building of SL_N over $\mathbb{Q}$, i.e. the simplicial set obtained as the nerve of the ordered set of non-zero proper linear subspaces of $\mathbb{Q}^N$. The Solomon–Tits theorem says that T_N is homotopy equivalent to wedge of $(N-2)$-spheres, see Theorem IV.5.2 of [8]. Thus the reduced homology $\tilde{H}_q(T_N; \mathbb{Z})$ of T_N with integral coefficients is zero except when $q = N-2$, in which case

$$\tilde{H}_{N-2}(T_N; \mathbb{Z}) =: \mathrm{St}_N$$

is by definition the *Steinberg module*. According to Proposition 1 of [50], the relative homology groups $H_q(X^*_N, \partial X^*_N; \mathbb{Z})$ are zero except when $q = N-1$, and

$$H_{N-1}(X^*_N, \partial X^*_N; \mathbb{Z}) = \mathrm{St}_N\,.$$

From this it follows that, for all $m \in \mathbb{N}$,

$$H^\Gamma_m(X^*_N, \partial X^*_N; \mathbb{Z}) \cong H_{m-N+1}(\Gamma; \mathrm{St}_N) \tag{3}$$

(see e.g. §3.1 of [50]). Combining this equality with the previous sections, we obtain:

Proposition 1.5. *For arbitrary positive integers $N > 1$ and m. We have the following isomorphism modulo $\mathcal{S}_{N+1}$*

$$H_{m-N+1}(\Gamma; \mathrm{St}_N) \cong H_m(\mathrm{Vor}_\Gamma) \quad \bmod \mathcal{S}_{N+1}. \tag{4}$$

2. Small cells of quadratic forms

The method described in [39], Sections 9.2 and 9.3, though not very efficient, can be used to classify cells of dimension $\frac{n(n+1)}{2} - t$ for small values of t with explicit calculations; these are the "small cells" referred to in the title and will allow us to give an explicit enumeration of the lowest dimensional cells of the Voronoi complexes associated to Γ which will be used in §4.

2.1. **Minimal classes and the perfection rank.** Our aim is the study of the Voronoi complex in a given (cellular) dimension n. We shall make use of Watson's index theory, a theory which is better understood in terms of lattices. For this reason we first recall some data of the "dictionary" which links lattices with quadratic forms. We refer to [40] for further details.

We identify a quadratic form q with the $n \times n$ symmetric matrix A such that $q(x) = X^t AX$, where X is the column-vector of the components of $x \in \mathbb{R}^n$. Let E be an n-dimensional Euclidean space, on which the *norm of x is $N(x) = x \cdot x$*. With a lattice $\Lambda \subset E$ (discrete subgroup of E of rank n) and a basis $\mathcal{B} = (e_1, \ldots, e_n)$ for Λ over $\mathbb{Z}$, we associate the Gram matrix $A = (e_i \cdot e_j)$ of $\mathcal{B}$ and the corresponding positive definite quadratic form q. The *minimum* of Λ and its set $m(\Lambda)$ of *minimal vectors* correspond to the same notions for quadratic forms. We denote by $s(\Lambda)$ the number of *pairs of minimal vectors* of Λ (or of q). Besides this "kissing number" an important invariant is the *perfection rank* r, the definition of which we recall now. Let $u \in \mathrm{End}(E)$. The transpose of u is the endomorphism ${}^t u$ such that $u(x) \cdot y = x \cdot {}^t u(y)$ for all $x, y \in E$. We say that u is symmetric if ${}^t u = u$. Let $\mathrm{End}^s(E)$ be the space of all symmetric endomorphisms. Given a line $D \subset E$, we denote by $p_D \in \mathrm{End}^s(E)$ the orthogonal projection to D, and write p_x if $D = \mathbb{R}\, x$ for some $x \neq 0$ in E. Note the formulae

$$p_x(y) = \frac{x \cdot y}{x \cdot x}\, x \quad \text{and} \quad \mathrm{Mat}(\mathcal{B}^*, \mathcal{B}, p_x) = XX^t$$

where $\mathcal{B}^*$ is the dual basis to $\mathcal{B}$, defined by $e_i \cdot e_j^* = \delta_{i,j}$. We denote by $\mathrm{Sym}_n(\mathbb{R})$ the real vector space of $n \times n$ symmetric matrices with coefficients in $\mathbb{R}$.

Definition 2.1. The *perfection rank of a family $D_1, \ldots, D_\ell$ (ℓ a positive integer) of lines in E* is the dimension of the span in $\mathrm{End}^s(E)$ of the projections p_{D_i}. A *perfection relation on the set $\{D_i\}$* is a non-trivial $\mathbb{R}$-linear relation of the form $\sum \lambda_i\, p_{D_i} = 0$. The *perfection rank r of a lattice* Λ is the perfection rank of the set of lines $\mathbb{R}\, x$, $x \in m(\Lambda)$; that of a *quadratic form q* is the rank in $\mathrm{Sym}_n(\mathbb{R})$ of the set $\{XX^t\}$, $X \in m(q)$.

We partition the space $\mathcal{L}$ of lattices into *minimal classes* by the relation $\Lambda \sim \Lambda'$ if and only if there exists $u \in GL(E)$ such that $u(\Lambda) = \Lambda'$ and $u(m(\Lambda)) = m(\Lambda')$, and order the minimal classes by the relation $C \prec C'$ if and only if there exists $\Lambda \in C$ and $\Lambda' \in C'$ such that $m(\Lambda) \subset m(\Lambda')$.

The dictionary above establishes a one-to-one correspondence between the set of minimal classes on the one hand, and the set of cells up to equivalence of (positive, definite) quadratic forms having a given minimum on the other hand. These are finite sets.

In the sequel, we restrict ourselves to *well-rounded lattices* (or *forms*), those for which the minimal vectors span E. The following proposition provides an easy test to decide whether two lattices belong to the same minimal class. Given a lattice Λ and a basis $\mathcal{B}$ for Λ, and a set S' of representatives of pairs of minimal vectors of

$m(\Lambda)$, let T be the $n \times s(\Lambda)$ matrix of the components of the vectors of S' on $\mathcal{B}$, and let $B = T\,T^t$. This is the *barycenter matrix of* $(\Lambda, \mathcal{B})$. Its equivalence class under $GL_n(\mathbb{Z})$ does not depend on $\mathcal{B}$.

Proposition 2.2. *Two lattices belong to the same minimal class if and only if their barycenter matrices define the same class under* $GL_n(\mathbb{Z})$*.*

Proof. This is Proposition 9.7.2 of [40]. □

Lemma 2.3. *Any perfection relation in* $\mathrm{End}^s(E)$ *between projections to vectors of* E *may be written in the form*

$$\textstyle\sum_{x\in S}\ \lambda_x p_x = \sum_{y\in T}\ \mu_y p_y$$

where λ_x, μ_y *are strictly positive and* S *and* T *span the same subspace of* E*.*

Proof. Getting rid of the zero coefficients, we obtain a relation of this kind for convenient subsets S, T of E, and we may moreover assume that the vectors x, y have norm 1. Applying this relation on a vector $z \in T^\perp$ and taking the scalar products with z, we obtain the equality

$$\textstyle\sum_{x\in S}\ \lambda_x\,(x\cdot z)^2 = 0$$

which shows that z also belongs to $S^\perp$. We have thus proved the inclusion $T^\perp \subset S^\perp$, i.e., $\langle S\rangle \subset \langle T\rangle$, and exchanging S and T shows that $\langle S\rangle = \langle T\rangle$. □

***Remark* 2.4.** By the lemma above, a perfection relation with non-zero coefficients between vectors which span an m-dimensional subset F of E involves at least $2m$ vectors. This is optimal for all $m \geqslant 2$, as shown by the union of two orthogonal bases $\mathcal{B}$ and $\mathcal{B}'$ for F, since the sum of the orthogonal projections to the vectors of $\mathcal{B}$ and $\mathcal{B}'$ both add to the orthogonal projection to F. This construction of perfection relations accounts for those of the lattice $\mathbb{D}_4$, since the set of minimal vectors $S(\mathbb{D}_4)$ is the union of three orthogonal frames.

Theorem 2.5. *The perfection rank* r *of a well-rounded* n*-dimensional lattice with kissing number* $s \leqslant n+5$ *is equal to* s*.*

Proof. By definition of the perfection rank we have $r \leqslant s$, and if $r < s$, there exists a perfection relation with support $2m$ vectors of some m-dimensional subspace F of E. We have $s \geqslant n+m$, hence $m \leqslant 5$. Now for a lattice L of dimension $n \leqslant 5$, one has $s = n$ except if $L \sim \mathbb{D}_4$ or if $n = 5$ and L has a $\mathbb{D}_4$-section having the same minimum. Since $s(\mathbb{D}_4) = 12$, we then have $s \geqslant n+8$, a contradiction. □

Notice that in the above theorem the bound $s - n \leqslant 5$ is optimal; see Example 2.7 below.

2.2. Watson's index theory and very small cells.

2.2.1. *Codes associated with well-rounded lattices.* Let Λ be a well-rounded lattice, let $e_1, \dots, e_n$ be n independent minimal vectors of Λ, and let Λ' be the sublattice of Λ generated by the e_i. Let γ_n be the Hermite constant in dimension n [40](§2.2). Then the index $[\Lambda : \Lambda']$ is bounded from above by $\gamma_n^{n/2}$ and so is the annihilator d of Λ/Λ'. The *maximal index* $\imath(\Lambda)$ of a well-rounded lattice Λ is the largest possible value of $[\Lambda : \Lambda']$ for a pair (Λ, Λ') as above.

Every element of Λ can be written in the form

$$x = \frac{a_1 e_1 + \cdots + a_n e_n}{d},\ a_i \in \mathbb{Z}\,,$$

and if $d > 1$ the systems $(a_1, \dots, a_n)$ mod d can be viewed as the words of a $\mathbb{Z}/d\mathbb{Z}$-code. The codes arising this way have been classified for n up to 8 in [39] (which relies on previous work by Watson, Ryshkov and Zahareva) and for $n = 9$ in [30]. The paper [39] relied on calculations which where feasible essentially by hand (together with some checks made using PARI [25]). This is no longer possible beyond dimension 8, and indeed [30] needed the use of a linear programming package, which was implemented on MAGMA [7].

By averaging on the automorphism group of the code (see Proposition 8.5 of [39]), one proves that if some code C of length n can be lifted to a pair (Λ, Λ') (we then say that C is *admissible*), then there exist Λ and Λ' which are invariant under $\mathrm{Aut}(C)$. Then the minimum $s_{\min}$ of s (over well-rounded lattices) is attained on such a lattice Λ, and the minimal class of Λ depends uniquely on C.

By inspection of the tables of [39](Tableau 11.1) and [30](Tables 2-10), one proves:

Proposition 2.6. *Let $d \geqslant 2$ and $n \leqslant 9$, and let C be an admissible $\mathbb{Z}/d\mathbb{Z}$-code. Then either C can be lifted to a pair (Λ, Λ') with $m(\Lambda) = \{\pm e_i\}$, or we have $s(\Lambda) \geqslant n + 6$ for every lift of C.* □

Before going further we give some more precise results on the index theory. The most useful tool is *Watson's identity*, relying vectors $e_1, \dots, e_n$ of a basis for E, a vector $e = \frac{a_1 e_1 + \cdots + a_n e_n}{d}$, and the vectors $e'_i := e - \mathrm{sgn}(a_i) e_i$:

$$\left(\Big(\sum_{i=1}^{n} |a_i| \Big) - 2d \right) N(e) = \sum_{i=1}^{n} |a_i| (N(e'_i) - N(e_i))\,.$$

Applied to minimal vectors e_i of a lattice $\Lambda = \langle e_i, e \rangle$, this proves the lower bound $\sum |a_i| \geqslant 2d$, and moreover shows that the vectors e'_i for which $a_i \neq 0$ are minimal whenever $\sum |a_i| = 2d$.

***Example* 2.7.** Let $n = 6$, $d = 3$ and let $(e_1, \dots, e_6)$ be a basis for E with $N(e_i) = 1$ and constant scalar products $e_i \cdot e_j = t$. Let $\Lambda = \langle e_1, \dots, e_6, e \rangle$ where $e = \frac{e_1 + \cdots + e_6}{3}$. Then for $\frac{1}{10} < t < \frac{1}{4}$ (e.g., $t = \frac{1}{5}$), the minimum of Λ is 1, $m(\Lambda) = \{\pm e_i, \pm e'_i\}$, and the perfection relations are proportional to $\sum p_{e_i} = \sum p_{e'_i}$, so that $s(\Lambda) = 12 = n + 6$ and $r = 11 = s - 1$. This is a consequence of the fact that we can associate canonically a perfection relation with every Watson identity [4](Proposition 2.5).

2.2.2. *Primitive minimal classes (or cells).* Let C be a minimal class. For every lattice $\Lambda \in C$, $\widetilde{\Lambda} = \Lambda \perp m\mathbb{Z}$, where m =minimum of Λ, is a lattice in $E \times \mathbb{R}$, which defines a minimal class $C' = C + \mathbb{Z}$ of dimension $n + 1$, containing all direct sums $\Lambda \oplus m\mathbb{Z}$ close enough to $\widetilde{\Lambda}$. We say that C is *primitive* if it does not extend by this process a class of dimension $n - 1$.

Among n-dimensional (well-rounded) minimal classes, that of $\mathbb{Z}^n$, which has $\imath = 1$ and $s(\Lambda) - n = 0$, plays a special role. Indeed, let C be a minimal class and let $e_1, \dots, e_n$ be independent minimal vectors of some lattice $\Lambda \in C$, and let $\Lambda' \subset \Lambda$ be the lattice with basis $(e_1, \dots, e_n)$. Let $I_1 \subset \{1, \dots, n\}$ be the support of the code defined by (Λ, Λ') ($I_1 = \emptyset$ if $\Lambda = \Lambda'$) and let I_2 be the set of subscripts which occur as components of minimal vectors distinct from the $\pm e_i$ ($I_2 = \emptyset$ if $s(\Lambda) = n$). Set $I = I_1 \cup I_2$ and $m = |I|$. If C is not the minimal class of $\mathbb{Z}^n$, then I is not empty and C extends a minimal class of dimension $m \geqslant 2$.

Therefore, to list minimal classes up to equivalence it suffices to list those which are primitive and then complete the list with those of the form $C' \oplus \mathbb{Z}$ for some class C' of dimension $n-1$.

2.2.3. *Classes with $s(\Lambda) = n$.*

Theorem 2.8. *The numbers of minimal classes of well-rounded lattices Λ with $s(\Lambda) = n$ and $n \leqslant 9$ having a given index $\imath$ are displayed in Table 2.*

Proof. The lattices Λ with $s(\Lambda) = n$ contain a unique sublattice Λ' generated by minimal vectors of Λ, and Λ' has a unique basis up to permutation and changes of signs. Hence the classification of minimal classes coincide with the index classification. We mainly need to consider the results from [39] and [30]. According to the previous section, for index 2 (resp. 3) there is one primitive class if $n \geqslant 5$ (resp. $n \geqslant 7$), hence $n-4$ (resp. $n-6$) cells. Similarly for cyclic quotients of order 4 and $n \geqslant 9$ there are $n-4$ primitive cells, but only three if $n = 8$, one if $n = 7$ and none if $n \leqslant 6$, which for $n = 7, 8, 9$ yields 1, 4, and 9 cells of cyclic type with $\imath = 4$. In the case $\imath = 4$ but with Λ/Λ' of type 2^2 (i.e. $\imath = 2^2$), the condition is that the corresponding binary code must be of weight $w \geqslant 5$. This implies $\ell \geqslant 8$, and if $\ell = 8$ (resp. $\ell = 9$), we are left with one code, with weight distribution $(5^2, 6)$ (resp. three codes, with weight distributions $(5^2, 8)$, $(5, 6, 7)$ and (6^3)). As a result we got four primitive cells. Finally we just have to read in the tables of [39](Tableau 11.1) and [30](Tables 2-10) for index $\imath \geqslant 5$ and $n \leqslant 9$ for all admissible codes for which $s(\Lambda) = n$ is possible. $\square$

$n \backslash \imath$	1	2	3	4	2^2	5	6	7	8	$4 \cdot 2$	# minimal classes
$\leqslant 4$	1	0	0	0	0	0	0	0	0	0	1
5	1	1	0	0	0	0	0	0	0	0	2
6	1	2	0	0	0	0	0	0	0	0	3
7	1	3	1	1	0	0	0	0	0	0	6
8	1	4	2	4	1	1	0	0	0	0	13
9	1	5	3	9	4	4	9	3	3	3	44

TABLE 2. Number of minimal classes of well-rounded lattices Λ with $s(\Lambda) = n$ and $n \leqslant 9$ according to the quotient.

To deal with slightly larger values of $s(\Lambda)$, we first establish an as short as possible list of *a priori* possible index systems, then test for minimal equivalence the putative classes obtained from this list, and finally construct explicitly lattices in the putative class or prove that such a class does not exist. This third step is the more difficult and it is hopeless to deal with relatively large values of $s(\Lambda)$ or of n without using efficient linear programming methods, as seen in §3.

2.3. **Lattices Λ with $s(\Lambda) = 1$ or $s(\Lambda) = 2$.** Consider a lattice Λ_0 having a basis of minimal vectors $\mathcal{B} = (e_1, \ldots, e_n)$ and $t = s(\Lambda_0) - n$ other minimal vectors $x_i = \sum_{i=1}^{n} a_{i,j} e_j$, $1 \leqslant j \leqslant t$. The $m \times m$ determinants ($m \leqslant \min(t, n)$) extracted from the matrix $(a_{i,j})$ are the *characteristic determinants* of Korkine and Zolotarev (cf. Chapter 6 from [40]). With every characteristic determinant $d \neq 0$ they associate a lattice Λ_0' of index $|d|$ in Λ_0. This shows that if $\imath(\Lambda_0) = 1$, then $d = 0$ or ± 1; in particular all $a_{i,j}$ are 0 or ± 1.

3. Enumeration of configurations of vectors

In this section we explain the algorithms used for enumerating the configurations of vectors in dimension $n \in \{8, \dots, 12\}$ and rank $r = n$, $n + 1$ or above.

Our approach computes first the configurations of vectors in rank $r = n$ and then from this enumeration gets the configurations in rank $r = n + 1$, then $n + 2$ and so on. The number of cases explodes with the dimension n and rank r as one expects and the computation is thus quite slow. However, for the case $r = n$ an additional problem occurs: the known bounds on the determinant of vector configurations are suboptimal. All computations rely on the ability to test if a configuration of shortest vectors of a positive definite matrix can be derived from a given configuration of vectors.

3.1. Testing realizability of vector family.

In [30] an algorithm for testing realizability of vector families by solving linear programs was introduced. We describe below the needed improvements of our strategy in order to reach higher dimensions. We refer to [45] for an account of the classical theory of linear programming.

Given m affine functions $(\phi_i)_{1 \leqslant i \leqslant m}$ on $\mathbb{R}^n$ and another (linear) function ϕ the *linear program* is to minimize $\phi(x)$ for x subject to the constraints $\phi_i(x) \geqslant 0$. Define $\mathcal{P} = \{x \in \mathbb{R}^n, \text{ s.t. } \phi_i(x) \geqslant 0\}$. The linear program is called *feasible* if $\mathcal{P} \neq \emptyset$ and the elements of $\mathcal{P}$ are called *feasible solutions*. If ϕ is bounded from below on $\mathcal{P}$ then the inferior limit is denoted $opt(\mathcal{P}, \phi)$ and is attained by one feasible solution. Any feasible solution will satisfy $\phi(x) \geqslant opt(\mathcal{P}, \phi)$

The *dual problem* is to maximize the value of m such that there exist β_i with

$$\phi = m + \sum_{i=1}^{m} \beta_i \phi_i \text{ with } \beta_i \geqslant 0$$

Any feasible solution of the dual problem will gives us $\phi(x) \geqslant m$ and the maximum value of such m will be exactly $opt(\mathcal{P}, \phi)$ by a theorem of von Neumann [45](§7.4). In other words, any feasible solution (m, β_i) of the dual problem will give us $opt(\mathcal{P}, \phi)$.

Let A be an $n \times n$ matrix with real coefficients and set $A[v] := v^t A v$ for any $v \in \mathbb{R}^n$. Given a configuration of vectors $\mathcal{V}$ the basic linear program to be considered is

$$\begin{aligned} \text{minimize} \quad & \lambda \\ \text{with} \quad & \lambda = A[v] \text{ for } v \in \mathcal{V} \\ & A[v] \geqslant 1 \text{ for } v \in \mathbb{Z}^n - \{0\} - \mathcal{V} \end{aligned}$$

If the optimal value satisfies $\lambda_{opt} < 1$ then $\mathcal{V}$ is realizable, otherwise not.

The main issue is that the above linear program has an infinity of defining inequalities and so instead we consider the program restricted to a finite subset, i.e. the linear inequalities:

$$A[v] \geqslant 1 \text{ for } v \in \mathcal{S} \text{ with } \mathcal{S} \text{ finite and } \mathcal{S} \subset \mathbb{Z}^n - \{0\} - \mathcal{V}.$$

It can happen that the equalities $\lambda = A[v]$ for $v \in \mathcal{V}$ have no solution with $\lambda \neq 0$. In that case $\mathcal{V}$ is not realizable.

It can also happen that the linear program is unbounded, that is, solutions with arbitrarily negative value of λ are feasible. In that case we append $2\mathcal{V}$ to $\mathcal{S}$.

Thus if those restrictions are implemented then the linear program has an optimal rational solution $A_{opt}(\mathcal{S})$ of optimal value $\lambda_{opt}(\mathcal{S})$.

According to the solution of the linear program we can derive following conclusions:

1. If $\lambda_{opt}(\mathcal{S}) \geqslant 1$ then we can conclude that the vector configuration is not realizable.
2. If $\lambda_{opt}(\mathcal{S}) < 1$ and $A_{opt}(\mathcal{S})$ is positive definite then we compute $\mathrm{Min}(A_{opt}(\mathcal{S}))$.
   (a) If $\mathrm{Min}(A_{opt}(\mathcal{S})) = \mathcal{V}$ then the configuration is realizable
   (b) Otherwise we cannot conclude. But we can insert the vectors in the difference $\mathrm{Min}(A_{opt}(\mathcal{S})) - \mathcal{V}$ into $\mathcal{S}$ and iterate.
3. If $\lambda_{opt}(\mathcal{S}) < 1$ and $A_{opt}(\mathcal{S})$ is not of full rank then we can compute some integer vector in the kernel of $A_{opt}(\mathcal{S})$ and insert them into $\mathcal{S}$ and iterate.
4. If $\lambda_{opt}(\mathcal{S}) < 1$ and $A_{opt}(\mathcal{S})$ is of full rank but not positive semidefinite then we can compute an integer vector v such that $A_{opt}(\mathcal{S})[v] < \lambda_{opt}(\mathcal{S})$ and insert it into $\mathcal{S}$ and iterate.

Thus we can iterate until we obtain either feasibility of the vector configuration or unfeasibility. In practice a naive implementation of this algorithm can be very slow and we need to apply a number of improvements in order to get reasonable running time:

1. The dimension of the program is $n(n+1)/2 - r$ and this is quite large. We can use symmetries in order to get smaller program. Namely we compute the group of integral linear transformation preserving $\mathcal{V}$ and impose that the matrix A also satisfies this invariance.
2. Even after symmetry reduction the linear programs have many inequalities and are hard to solve. In our implementation we use `cdd` [22] based on exact arithmetic and provides solutions of the linear program and its dual in exact rational arithmetic. However, `cdd` uses the simplex algorithm and is very slow in some cases. Thus the idea is to use floating point arithmetic and the `glpk` program [24] which has better algorithms and can solve linear programs in double precision. From the approximate solution we can guess in most cases a feasible rational solution of the linear program and its dual. If both gives the same value, then we have resolved our linear program. If this approach fails, then we fall back to the more expensive in time `cdd`. In all cases, we only accept a solution if it has a corresponding dual solution.
3. If the matrix $A_{opt}(\mathcal{S})$ is of full rank but not positive definite then there exists an eigenvector $w \in \mathbb{R}^n$ of eigenvalue $\alpha < 0$. We then use the sequence of vectors
$$w^i = (Near(iw_1), Near(iw_2), \dots, Near(iw_n)).$$
   with $Near(x)$ being the nearest integer to a real number x. As i increases w^i approaches the direction of the vector w. Thus there is an index i_0 such that $w^{i_0} \neq 0$ and $A_{opt}(\mathcal{S})[w^{i_0}] < \lambda_{opt}(\mathcal{S})$ and this vector w^{i_0} can be inserted into $\mathcal{S}$. The problem is that in many cases the matrix $A_{opt}(\mathcal{S})$ is very near to being positive definite and the negative eigenvalue will be very small. Thus we first try double precision with the `Eigen` template matrix library [27] for the computation of the eigenvector and vector w^{i_0}. If this fails then we use the arbitrary precision floating point library `mpfr` [21], still with `Eigen` and progressively increase the number of digits until a solution is found.

4. An issue is for the initial vector set. In our implementation we set
$$\mathcal{S} = \mathcal{V} \cup \left\{ x \pm e_i \pm e_j \text{ for } x \in \mathcal{V} \text{ and } 1 \leqslant i, j \leqslant n \right\} - \{0\}$$
but there could be better choice for initial vector set.
5. We apply LLL reduction [13] to the vector configuration. Namely, we define a positive definite matrix
$$A_{\mathcal{V}} = \sum_{v \in \mathcal{V}} v^t v$$
and apply the LLL reduction to it in order to get a reduction matrix $P \in GL_n(\mathbb{Z})$. The matrix $P \in GL_n(\mathbb{Z})$ is then used to reduce $\mathcal{V}$ as well. The use of LLL reduction reduces the maximal size of the coefficients and dramatically reduces the number of iterations needed to get a result. Hence we use it systematically.

When all those methods are implemented we manage to do the realizability tests in reasonable time.

A variant of the above mentioned realizability algorithm is to consider a family of vector $\mathcal{V}$ of rank r and return a realizable configuration of vectors $\mathcal{W}$ of rank r if it exists that contains $\mathcal{V}$. It suffices in the case $\lambda_{opt}(\mathcal{S}) < 1$ and $A_{opt}(\mathcal{S})$ to distinguish between vectors of $\mathrm{Min}(A_{opt})$ that increases the rank and vectors that do not increase the rank.

3.2. **Enumeration of configurations of vectors with $r = n$.** In [30] it is proved that for a configuration $\{\pm v_1, \dots, \pm v_n\}$ of shortest vectors of a lattice, we have
$$|\det(v_1, \dots, v_n)| \leqslant \sqrt{\gamma_n^n}\,.$$
As it turns out this upper bound on the determinant is tight for dimension $n \leqslant 8$ but not in dimension 9 and 10. An additional problem is that γ_n is known exactly only for $n \leqslant 8$ and $n = 24$. Our strategy is thus to simply enumerate the vector configurations up to the best upper bound that we have on the determinant.

For dimension 10, combined with the known upper bound on γ_{10} this gives an upper bound of 59 on the indexes of the relevant lattices [30]. If one uses the conjectured value of γ_{10} then one gets 36 as upper bound in dimension 10. It will turn out that the maximal possible determinant is 16.

A key aspect of the enumeration is to enumerate first the cases where the quotient $\mathbb{Z}^n/L$ has the structure of a prime cyclic group. This is important since if a prime p is unfeasible then any vector configuration with determinant divisible by p is unfeasible as well. It is also important since our enumeration goes prime by prime for composite determinants. For $d = p_1 \times \cdots \times p_r$ we first do the enumeration of vector configurations of determinant p_1, then $p_1 p_2$ and so on.

In the case of lattices of index p with p prime we consider a lattice L spanned by $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ and
$$e_{n+1} = \frac{1}{p}(a_1, \dots, a_n), a_i \in \mathbb{Z}$$
such that $(e_1, \dots, e_n)$ is the configuration of shortest vectors of a lattice. By standard reductions, we can assume that

- $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$
- and $1 \leqslant a_i \leqslant \lfloor p/2 \rfloor$.

Since p is prime e_{n+1} can be replaced by ke_{n+1} for any $1 \leqslant k \leqslant p-1$. Thus we can assume the vector e_{n+1} to be lexicographically minimal among all possible vectors.

It turns out that lexicographically minimal vector configurations can be enumerated by exhaustive enumeration without having to store in memory the list of candidates. The idea is as follows: if $(a_1, \ldots, a_n)$ is lexicographically minimal, then $(a_1, \ldots, a_{n-1})$ is also lexicographically minimal. Lexicographic minimality only requires $(p-1)n$ multiplications and reductions to be tested. Thus we enumerate all configuration up to length $n-1$ and then extend this enumeration to length n by adding all possible feasible candidates. For $n = 10$ and $p = 59$ we have 16301164 possible vector configurations and for each of them we test realizability.

When the enumeration for index p is done we can continue the enumeration up to index pp' by taking all feasible lattices of index p and considering all their sublattices of index p' up to action of the symmetry group. Thus we get a set of *a priori* feasible lattices for which we can apply our realizability algorithm and get a list of lattices of index pp'. For $n = 10$ the most complex case of this kind is $49 = 7^2$.

By doing prime by prime up to 59 in this way we are able to get all configurations of shortest vectors in dimension 10, we find 283 different lattices. For dimension $n = 11$, we were only able to go up to index 45 and we got in total 6674 possible sublattices. The list is not proved to be complete but it is reasonable to conjecture that this list is complete since the maximum determinant of a realizable vector configuration is 32. For dimension $n = 12$, we managed only to go up to index 30 and found 454576 different vector configurations and it seems that this list is far from complete.

3.3. **Enumeration of configurations of vectors with $r > n$.** For the case $r = n+1$ and $r = n+2$ we have proved in [14] that the relevant cones are simplicial. In [14] an algorithm is given for getting the full list of configuration vectors in those ranks. The only improvement to this algorithm is that the integer points are obtained by an exhaustive enumeration procedure since `zsolve` proved too slow.

For rank $r = n+3$ and above simpliciality is not *a priori* true, though it is expected to hold in rank $n+3$ and $n+4$. Thus we need a different approach to the enumeration. If we have a configuration of vectors $\mathcal{V}'$ in dimension n and rank $r > n+2$ then it necessarily contains a n-dimensional configurations $\mathcal{V}$ of rank $r-2$ and two n-dimensional configurations $\mathcal{W}_1$ and $\mathcal{W}_2$ of rank $r-1$ such that

$$\mathcal{V} \subset \mathcal{W}_i \subset \mathcal{V}'.$$

Our approach is as follows:

1. We first enumerate the configurations of vectors in dimension n and rank $r-2$ and $r-1$.
2. We determine all the orbits of pairs $(\mathcal{V}, \mathcal{W})$ with $\mathcal{V}$ of dimension n rank $r-2$, $\mathcal{W}$ of dimension n rank $r-1$.
3. For any configuration $\mathcal{V}$ of dimension n rank $r-2$ there is a finite number of configurations of vectors of dimension n rank $r-1$ containing it. We can thus enumerate all the pairs $(\mathcal{W}_1, \mathcal{W}_2)$ containing $\mathcal{V}$ and check if $\mathcal{W}_1 \cup \mathcal{W}_2$ is contained in a realizable family of vectors.

3.4. **Obtained enumeration results.** By combining all above enumeration methods, we can obtain the number of orbits of perfect domains for small r and n.

Proposition 3.1. *The number of orbits of cones in the perfect cone decomposition for rank $r \leqslant 12$ and dimension n at most 11 (the result for $r = n = 11$ is conjectural) are given in Table 3.*

$r\backslash n$	4	5	6	7	8	9	10	11	12
4	1	3	4	4	2	2	2	-	-
5		2	5	10	16	23	25	23	16
6			3	10	28	71	162	329	589
7				6	28	115	467	1882	7375
8					13	106	783	6167	50645
9						44	759	13437	?
10							283	16062	?
11								6674?	?

Table 3. Known number of orbits of cones in the perfect cone decomposition for rank $r \leqslant 12$ and dimension n at most 11 (*the result for $r = n = 11$ is conjectural*).

***Remark* 3.2.** For dimension $n = 11, 12$ we do not have a full enumeration, however the partial configurations obtained are already instructive. In all dimensions $n \leqslant 10$ the orientations of the well-rounded families of vectors with $s = n$ (i.e., the orientation of the associated cells of dimension n in X_n^*) were found not to be preserved by their stabilizer. However, this changes with 5 well-rounded configurations known in dimension 11 and 12 in dimension 12. One such configuration in dimension 11 is $e_5 + e_6 + e_7 + e_{10} - e_3$, $e_{11} - e_2 - e_{10}$, $e_3 + e_9$, $e_4 + e_6 + e_8 - e_2 - e_9$, $e_1 + e_2 + e_3$, $e_4 + e_7 + e_{11} - e_6$, $e_8 + e_9 - e_7$, $e_1 + e_5 - e_{11}$, $e_4 + e_5$, $e_1 - e_8$, e_{10} and it has a stabilizer of order 4 which is the minimum known so far. It seems reasonable to expect that there are well-rounded vector configurations with stabilizer of order 2, i.e. only antipodal operation. We know just one well-rounded configuration in dimension 12 whose orbit under $GL_{12}(\mathbb{Z})$ splits into two orbits under $SL_{12}(\mathbb{Z})$. One representative is $e_6 - e_1 - e_2 - e_3 - e_4$, $e_6 - e_7 - e_8$, $e_9 - e_3 - e_6 - e_{10} - e_{11} - e_{12}$, $e_7 + e_{12} - e_1 - e_2 - e_5 - e_8$, $e_{11} - e_1 - e_4 - e_5 - e_6 - e_{10}$, e_2, $e_4 + e_8 - e_7$, e_9, e_{10}, $e_3 + e_5 + e_7 + e_9 - e_1$, e_{11}, e_{12}.

***Remark* 3.3.** The above data for $r \leqslant 7$ recover the computations of [19] (cf. Figures 1 and 2) and [38].

4. Homology of the Voronoi complexes

4.1. **Preliminaries.** Recall the following simple fact, cf. p.602 of [19], which are relevant for understanding the action of $GL_N(\mathbb{Z})$ on X_N:

Lemma 4.1.
- *Assume that p is a prime and $g \in GL_N(\mathbb{R})$ has order p. Then $p \leqslant N + 1$.*
- *The action of $GL_N(\mathbb{R})$ on the symmetric space X_N preserves its orientation if and only if N is odd.*

***Remark* 4.2.** We can give a more precise statement regarding the torsion in $GL_N(\mathbb{Z})$. Let p an odd prime and k a positive integer. Set $\psi(p^k) = \varphi(p^k)$, $\psi(2^k) = \varphi(2^k)$ if $k > 1$ and set $\psi(2) = \psi(1) = 0$. For an arbitrary $m = \prod_\alpha p_\alpha^{k_\alpha}$ define $\psi(m) = \sum_\alpha \psi(p_\alpha^{k_\alpha})$.

According to the crystallographic restriction theorem, an elementary proof of which is given as Theorem 2.7 of [35], an element of order m occurs in $GL_N(\mathbb{Z})$ if and only if $\psi(m) \leqslant N$.

4.2. **The Voronoi complexes in low dimensions.** From the computations of §2 and §3, we deduce the following cardinalities.

Proposition 4.3. *The number of low dimensional cells in the quotient* $(X_N^*, \partial X_N^*)/GL_N(\mathbb{Z})$ *is given by Table 4.*

n	8	9	10	11	12
$\Sigma_n^\star(GL_8(\mathbb{Z}))$	13	106	783	6167	50645
$\Sigma_n\,(GL_8(\mathbb{Z}))$	0	0	0	0	0
$\Sigma_n^\star(GL_9(\mathbb{Z}))$		44	759	13437	?
$\Sigma_n\,(GL_9(\mathbb{Z}))$		0	0	0	?
$\Sigma_n^\star(GL_{10}(\mathbb{Z}))$			283	16062	?
$\Sigma_n\,(GL_{10}(\mathbb{Z}))$			0	0	?

Table 4. Cardinality of Σ_n and $\Sigma_n^\star$ for $N = 8, 9, 10$ (empty slots denote zero).

The explicit data can be retrieved at the url https://github.com/elbazvip/Voronoi-complexes-database. For convenience to the reader, we give below a set of representatives in the case $s(\Lambda) = n = 8$.

Proposition 4.4. *A set of representatives for the 13 well-rounded cells of* $\Sigma_8^\star(GL_8(\mathbb{Z}))$ *is given by the following matrices:*

$$\begin{pmatrix} 0&0&0&0&0&-1&0&0\\ 0&-1&0&0&0&1&0&0\\ 0&-1&0&1&0&1&1&0\\ 0&0&0&1&0&1&0&0\\ 0&-1&-1&1&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 0&2&0&-2&0&-1&-1&-1\\ 0&-1&-1&1&1&1&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&1&1&0&0&0\\ 0&0&-1&-2&-1&0&0&0\\ 0&0&-2&-2&-2&0&-1&0\\ 0&0&-1&-1&-1&0&0&-1\\ 0&-1&-1&-1&-2&0&0&0\\ 1&1&1&1&1&0&0&0\\ -1&1&3&3&3&0&1&1\\ 0&-1&-2&-2&-2&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0&0&0&-1&0\\ 0&-1&-2&0&0&0&1&0\\ 0&-1&-2&0&0&0&1&1\\ 0&-1&-1&0&0&-1&0&0\\ 0&-1&-1&0&0&1&0&0\\ 1&1&1&0&0&0&0&0\\ 0&1&3&0&-1&0&0&0\\ -1&-1&-2&1&1&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&1&-1&0&0&0&-1\\ 0&0&-2&1&0&0&0&1\\ 0&-1&-2&1&0&1&0&1\\ 0&-1&-1&0&0&0&0&1\\ 0&-1&-1&1&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 0&2&3&-2&0&-1&-1&-2\\ 0&-1&-2&1&1&1&1&1 \end{pmatrix}, \begin{pmatrix} -1&-2&0&0&1&1&1&-1\\ 0&1&0&0&0&-1&-1&0\\ 0&1&0&0&-1&-1&-1&2\\ 0&0&1&0&0&0&0&-1\\ 0&-1&0&0&0&0&1&-1\\ 0&1&0&0&-1&-1&-2&1\\ 0&-1&-1&-1&1&1&1&0\\ 1&1&1&1&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&-1&0&0&0&1\\ 0&-1&0&1&0&0&1&0\\ 0&1&0&0&-1&0&-1&0\\ 1&-1&0&1&1&0&0&0\\ 0&1&-1&0&0&0&0&0\\ -1&1&1&0&0&0&0&0\\ 0&2&0&-1&-1&-1&-1&0\\ 1&-2&0&1&1&1&1&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0&-1&1&0&0\\ 0&0&1&-1&-1&0&0&1\\ 0&0&0&1&1&-1&0&-1\\ 0&0&0&0&0&0&1&1\\ 0&0&-1&0&0&-1&0&0\\ 1&0&1&0&0&0&0&0\\ 0&0&1&-1&-1&1&0&0\\ 0&1&-1&1&1&0&0&0 \end{pmatrix}, \begin{pmatrix} -1&1&-1&-2&-1&-1&-2&0\\ 0&0&1&1&0&0&1&0\\ -1&0&-1&-1&-1&0&-1&1\\ 0&-1&0&1&1&1&2&0\\ 1&0&2&1&1&0&1&0\\ 0&1&0&0&0&-1&-1&0\\ 0&1&-1&-1&-1&-1&-1&0\\ 1&0&1&1&1&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&1&-1&-1&-1&0&0&1\\ -1&0&1&1&1&-1&-1&0\\ 0&0&-1&0&0&1&1&0\\ -1&0&1&0&0&-1&0&0\\ 0&-1&1&1&0&0&0&0\\ 0&0&0&0&1&1&0&0\\ 1&0&-1&-1&0&0&0&0\\ 1&1&0&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 1&1&0&0&-1&0&0&0\\ -1&0&0&-1&1&0&-1&0\\ 1&-1&0&1&0&0&0&0\\ -1&0&0&-1&0&0&-1&1\\ -1&-1&0&0&0&1&1&0\\ 0&-1&-1&0&1&0&0&0\\ 1&1&0&1&0&0&0&0\\ 1&1&1&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1&0&-1\\ 0&0&0&1&0&1&0&1\\ 0&0&0&-1&0&-1&1&0\\ -1&0&1&-1&0&-2&0&0\\ 0&1&-1&0&1&1&0&0\\ 0&-1&1&1&0&0&0&0\\ 0&0&1&-1&0&-1&0&0\\ 1&1&-1&1&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&-1&0&0&1&0&-1&0\\ 0&1&-1&0&0&0&0&1\\ 0&-1&1&0&0&0&1&0\\ 1&1&-1&0&-1&1&0&0\\ -1&-1&1&1&0&0&0&0\\ -1&0&0&0&1&0&0&0\\ 1&2&-2&0&0&0&0&0\\ 0&-1&2&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&1&0&0&0&-1\\ -1&0&0&-1&0&0&1&0\\ 0&0&-1&-1&0&0&1&0\\ 0&0&-1&0&0&0&1&1\\ 0&-1&0&-1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 0&0&1&1&0&-1&-2&0\\ 0&0&-1&-1&1&1&1&0 \end{pmatrix}.$$

We can check the inequivalence of the above matrices using the command `qfisom` from PARI [25]. As all cells have their orientations changed, we deduce the following central result for the homology of the Voronoi complex.

Theorem 4.5. *The groups* $H_k(\mathrm{Vor}_{GL_N(\mathbb{Z})})$ *are zero modulo* $\mathcal{S}_2$ *for* $N = 8$ *and* $k \leqslant 12$*,* $N = 9, 10$ *and* $k < 12$*.*

***Remark* 4.6.** In the case $N = 12$ we cannot prove so far that the list of 12-dimensional cells of $\mathrm{Vor}_{GL_N(\mathbb{Z})}$ is complete (even if heuristically it seems the case).

5. Cohomology of modular groups

5.1. **Borel–Serre duality.** According to Borel and Serre, Thm. 11.4.4 and Thm. 11.5.1 of [6], the group $\Gamma = SL_N(\mathbb{Z})$ or $GL_N(\mathbb{Z})$ is a virtual duality group with dualizing module

$$H^{v(N)}(\Gamma; \mathbb{Z}[\Gamma]) = \mathrm{St}_N \otimes \tilde{\mathbb{Z}},$$

where $v(N) = N(N-1)/2$ is the virtual cohomological dimension of Γ and $\tilde{\mathbb{Z}}$ is the orientation module of X_N. It follows that there is a long exact sequence

$$\cdots \longrightarrow H^{v(N)-n}(\Gamma; \tilde{\mathbb{Z}}) \longrightarrow \hat{H}^{v(N)-n}(\Gamma, \tilde{\mathbb{Z}}) \longrightarrow H_{n-1}(\Gamma; \mathrm{St}_N) \longrightarrow H^{v(N)-n+1}(\Gamma; \tilde{\mathbb{Z}}) \longrightarrow \cdots . \tag{5}$$

where $\hat{H}^*$ is the Farrell cohomology of Γ [20].

From Lemma 4.1 and the Brown spectral sequence, X (4.1) of [8], we deduce that $\hat{H}^*(\Gamma, \tilde{\mathbb{Z}})$ lies in $\mathcal{S}_{N+1}$. Therefore

$$H_n(\Gamma; \mathrm{St}_N) \equiv H^{v(N)-n}(\Gamma; \tilde{\mathbb{Z}}), \text{ modulo } \mathcal{S}_{N+1}. \tag{6}$$

When N is odd, then $\mathrm{GL}_N(\mathbb{Z})$ is the product of $\mathrm{SL}_N(\mathbb{Z})$ by $\mathbb{Z}/2$, therefore

$$H^m(\mathrm{GL}_N(\mathbb{Z}); \mathbb{Z}) \equiv H^m(\mathrm{SL}_N(\mathbb{Z}); \mathbb{Z}), \text{ modulo } \mathcal{S}_2.$$

When N is even, then the action of $\mathrm{GL}_N(\mathbb{Z})$ on $\tilde{\mathbb{Z}}$ is given by the sign of the determinant (see Lemma 4.1) and Shapiro's lemma gives

$$H^m(\mathrm{SL}_N(\mathbb{Z}), \mathbb{Z}) = H^m(\mathrm{GL}_N(\mathbb{Z}), M), \tag{7}$$

with

$$M = \mathrm{Ind}_{\mathrm{SL}_N(\mathbb{Z})}^{\mathrm{GL}_N(\mathbb{Z})} \mathbb{Z} \equiv \mathbb{Z} \oplus \tilde{\mathbb{Z}}, \text{ modulo } \mathcal{S}_2.$$

5.2. **Homology of modular groups with coefficients in the Steinberg module.** From 4.5 and 1.5, we deduce

Corollary 5.1. *The groups* $H_{k-N+1}(GL_N(\mathbb{Z}); \mathrm{St}_N)$ *are trivial modulo* $\mathcal{S}_{N+1}$ *for* $N = 8, 9, 10$ *and* $k = 8, \ldots, 11$ *(assuming* $k \geqslant N$*).*

Adding this to the results of [19], one obtains Table 5.

$k \backslash N$	0	1	2	3	4	5	6	7	8	9	10	11
12								$\mathbb{Z}$	?	?	?	?
11									?	?	?	?
10						$\mathbb{Z}$	$\mathbb{Z}$		?	?	?	?
9									?	?	?	?
8									?	?	?	?
7								$\mathbb{Z}$	?	?	?	?
6							$\mathbb{Z}$	$\mathbb{Z}$	?	?	?	?
5						$\mathbb{Z}$	$\mathbb{Z}$			?	?	?
4										?	?	?
3				$\mathbb{Z}$	$\mathbb{Z}$						?	?
2												?
1												
0	$\mathbb{Z}$	$\mathbb{Z}$										

TABLE 5. The groups $H_{k+N-1}(\mathrm{Vor}_{GL_N(\mathbb{Z})}) = H_k(GL_N(\mathbb{Z}); \mathrm{St}_N)$ modulo $\mathcal{S}_{N+1}$. Empty slots denote 0.

5.3. **Cohomology of modular groups.** When $\Gamma = SL_N(\mathbb{Z})$ or $GL_N(\mathbb{Z})$, we know $H^m(\Gamma; \tilde{\mathbb{Z}})$ by combining (4) (end of §1.2.3), Section 4.2 and (6). As shown above, this allows us to compute the cohomology of Γ with trivial coefficients. The results are given in Corollary 5.2 below.

Corollary 5.2. *From Borel–Serre duality, we have*

$$H^{\frac{N(N-1)}{2}-k}(GL_N(\mathbb{Z}); \mathbb{Z}) = 0 \mod \mathcal{S}_{N+1} ,$$

for $N = 8, 9, 10, 11$ *and* $0 < k \leqslant 12 - N$.

***Remark* 5.3.** This provides further evidence for a conjecture of Church, Farb and Putman [11], see Conjecture 2.

6. Application to algebraic K-theory of integers

The homology of the general linear group with coefficients in the Steinberg module can also be used to compute the K-theory of $\mathbb{Z}$. Let $P(\mathbb{Z})$ be the exact category of free $\mathbb{Z}$-modules of finite rank, let Q be the category obtained from $P(\mathbb{Z})$ by applying Quillen's Q-construction [44], and let BQ be its classifying space. Let Q_N be the full subcategory of Q containing all free $\mathbb{Z}$-modules of rank at most N and BQ_N its classifying space. One of the definitions of the algebraic K-theory groups [44] is

$$K_m(\mathbb{Z}) = \pi_{m+1}(BQ) , \quad m \geqslant 0 .$$

Therefore we can compute $K_m(\mathbb{Z})$ if we understand the homology of BQ as well as the Hurewicz map

$$h_m \colon K_m(\mathbb{Z}) \to H_{m+1}(BQ; \mathbb{Z}).$$

To do so, we use that Quillen proved in Theorem 3 of [43] that there are long exact sequences

$$\cdots \longrightarrow H_m(BQ_N;\mathbb{Z}) \longrightarrow H_{m-N}(GL_N(\mathbb{Z});\mathrm{St}_N) \longrightarrow H_{m-1}(BQ_{N-1};\mathbb{Z}) \longrightarrow H_{m-1}(BQ_N;\mathbb{Z}) \longrightarrow \cdots$$

Since $BQ_0 \simeq *$, these allow us to inductively obtain information about the homology of BQ_N from $H_*(GL_N(\mathbb{Z});\mathrm{St}_N)$.

6.1. **On the homology of BQ.** Using Proposition 1.5, we can rewrite the above Quillen sequences as

$$\cdots \longrightarrow H_m(BQ_N;\mathbb{Z}) \longrightarrow H_{m-1}(\mathrm{Vor}_{GL_N(\mathbb{Z})}) \longrightarrow H_{m-1}(BQ_{N-1};\mathbb{Z}) \longrightarrow H_{m-1}(BQ_N;\mathbb{Z}) \longrightarrow \cdots \tag{8}$$

We obtain the following result concerining the homology of BQ. It is proven using a spectral sequence, but a reader unfamiliar with spectral sequences may reproduce the results by inductively using the long exact sequences (8).

Proposition 6.1. *Modulo $\mathcal{S}_7$ we have*

$$H_m(BQ;\mathbb{Z}) = \begin{cases} 0 & \text{if } m = 2,3,4,5,8,9 \\ \mathbb{Z} & \text{if } m = 0,1,6,7,10,11. \end{cases}$$

Furthermore, modulo $\mathcal{S}_{11}$ we have $H_{12}(BQ;\mathbb{Z}) = \mathbb{Z}$.

Proof. The long exact sequences (8) give rise to a spectral sequence

$$E^1_{pq} = H_q(GL_p(\mathbb{Z});\mathrm{St}_p) \Longrightarrow H_{p+q}(BQ;\mathbb{Z}).$$

We shall use this modulo $\mathcal{S}_7$ and $\mathcal{S}_{11}$ respectively. Table 5 gives most of the E^1-page for $p \leqslant 11$ and $q \leqslant 12$, substituting $N = p$ and $k = q$.

For $m \leqslant 12$, the diagonal line $p + q = m$ contains either no non-zero entry, or a single entry $\mathbb{Z}$. Thus to prove the first part it suffices to show that the d^1-differentials $d^1 \colon E^1_{p+1,q} \to E^1_{p,q}$ vanish modulo $\mathcal{S}_7$ for $p + q \leqslant 12$. For the second part, we need to further verify that all d^r-differentials $d^r \colon E^r_{6+r,6-r+1} \to E^r_{6,6}$ vanish modulo $\mathcal{S}_{11}$. Since these are homomorphisms into abelian groups that are either zero or free, we can prove this by verifying it after tensoring with $\mathbb{Q}$.

To do so, we use that the rational homotopy groups of BQ are known by work of Borel, §12 of [5]:

$$\pi_m(BQ) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } m = 1 \text{ or, } m = 4i+2 \text{ with } i \text{ a positive integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Since BQ is an infinite loop space, the rational homotopy groups determine its rational homology groups: for $m \leqslant 12$, $H_m(BQ;\mathbb{Q})$ is 0 if $m = 2,3,4,5,8,9$ and $\mathbb{Q}$ if $m = 0,1,6,7,10,11,12$. This proves the desired statement. □

***Remark* 6.2.** That the coinvariants $H_0(GL_N(\mathbb{Z});\mathrm{St}_N)$ vanish for $N \geqslant 3$ is due to Lee and Szczarba, Theorem 1.3 of [37]. They deduce this by exhibiting a *generating set* of St_N. In [12], Church and Putman give a *presentation* of St_N, from which one may deduce that $H_1(GL_N(\mathbb{Z});\mathrm{St}_N) = 0$ modulo $\mathcal{S}_N$ for $N \geqslant 3$. In fact, Theorem

A of [12] gives only a rational statement, but it is straightforward to verify their argument goes through modulo $\mathcal{S}_N$. In [42], Miller, Patzt, and Nagpal have shown that $H_1(SL_N(\mathbb{Z}); \mathrm{St}_N) = 0$ for $N \geqslant 6$, that is, without needing to work modulo $\mathcal{S}_N$. More recently, Brück, Miller, Patzt, Sroka and Wilson [9] proved a similar vanishing result (rationally) for $H_2(SL_N(\mathbb{Z}); \mathrm{St}_N)$ and $N \geqslant 3$.

6.2. **On the Hurewicz homomorphism.** By definition, for every integer $m \geqslant 1$,

$$K_m(\mathbb{Z}) = \pi_{m+1}(BQ)\,.$$

The space BQ is in fact $\Omega^{\infty-1}\mathbf{K}(\mathbb{Z})$ with $\mathbf{K}(\mathbb{Z})$ the algebraic K-theory spectrum of $\mathbb{Z}$, so in particular an infinite loop space. This has consequences for the kernel C_m of the Hurewicz homomorphism

$$h_m \colon \pi_m(BQ) \to H_m(BQ; \mathbb{Z})\,.$$

Proposition 6.3. *Modulo $\mathcal{S}_5$, we have $C_m = 0$ for $m = 9, 10, 11$. Modulo $\mathcal{S}_7$, we have $C_{12} = 0$.*

Proof. This follows from Theorem 1.5 of [1], which implies that if X is a path-connected infinite loop space then the kernel of the Hurewicz homomorphism $\pi_n(X) \to H_n(X; \mathbb{Z})$ is annihilated by R_n, an integer divisible only by primes $\leqslant \frac{n}{2} + 1$. We apply this result to $X = BQ$. □

Theorem 6.4. *The group $K_8(\mathbb{Z})$ is trivial.*

Proof. From Propositions 6.1 and 6.3 we deduce that $K_8(\mathbb{Z}) = 0$ modulo $\mathcal{S}_7$. According to the *Quillen–Lichtenbaum conjectures* (see e.g. Chapter VI.10 of [53]) if ℓ is a regular odd prime, there are no ℓ-torsion in $K_{2j}(\mathbb{Z})$ for $j > 0$. Hence $K_8(\mathbb{Z}) = 0$. □

Using an elaboration of the method presented, we can recover information about several related algebraic K-theory groups (see also Table VI.10.1.1 of [53]).

Proposition 6.5. *Modulo $\mathcal{S}_7$, $K_9(\mathbb{Z}) \cong \mathbb{Z}$ and $K_{10}(\mathbb{Z}) = 0$. Modulo $\mathcal{S}_{11}$, $K_{11}(\mathbb{Z}) = 0$.*

Proof. Having proven Theorem 6.4, we know the groups $K_i(\mathbb{Z})$ modulo $\mathcal{S}_7$ for $i \leqslant 8$: they vanish unless $i = 0, 5$ in which case they are $\mathbb{Z}$. The groups $K_i(\mathbb{Z})$ are also the homotopy groups of the algebraic K-theory spectrum $\mathbf{K}(\mathbb{Z})$, and we conclude that there is a map of spectra

$$\mathbf{S}^0 \vee \mathbf{S}^5 \to \mathbf{K}(\mathbb{Z})$$

which is 8-connected modulo $\mathcal{S}_7$. By the Hurewicz theorem modulo $\mathcal{S}_7$, we see that

$$\pi_{10}(BQ) \cong H_{10}(BQ, \Omega^{\infty-1}(\mathbf{S}^0 \vee \mathbf{S}^5); \mathbb{Z})$$

modulo $\mathcal{S}_7$. It is a standard computation that $H_{10}(\Omega^{\infty-1}(\mathbf{S}^0 \vee \mathbf{S}^5); \mathbb{Z}) = 0$ modulo $\mathcal{S}_7$, so from Propositions 6.1 and the long exact sequence of a pair it follows that $K_9(\mathbb{Z}) = \pi_{10}(BQ) \cong \mathbb{Z}$ modulo $\mathcal{S}_7$.

This allows for the construction of a further map

$$\mathbf{S}^0 \vee \mathbf{S}^5 \vee \mathbf{S}^9 \to \mathbf{K}(\mathbb{Z})$$

which is 9-connected modulo $\mathcal{S}_7$. Applying $\Omega^{\infty-1}$ and repeating the above analysis in degrees 11 and 12 gives $K_{10}(\mathbb{Z}) = 0$ modulo $\mathcal{S}_7$ and $K_{11}(\mathbb{Z}) = 0$ modulo $\mathcal{S}_{11}$. □

***Remark* 6.6.** As pointed out in Remark 6.2, [12] proves that $H_1(GL_N(Z); \mathrm{St}_N)$ vanishes modulo $\mathcal{S}_N$. In order to prove $K_{12}(\mathbb{Z}) = 0$, we thus "only" need to recover the groups $H_{12}(\mathrm{Vor}_{GL_N(\mathbb{Z})})$ for $N = 9, 10, 11$ which are still missing.

7. Arithmetic applications

For the convenience of the reader, we recall some facts about the relationship between algebraic K-theory and étale cohomology, with a view towards the Kummer–Vandiver conjecture. We follow the presentation of Kurihara [34] and Soulé [49] (see also Section VI.10 of [53]).

Let p be an odd prime, $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. Denote by

$$H^i_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}_p(j)) := \varprojlim_{\nu} H^i_{\text{ét}}(\operatorname{Spec}(\mathbb{Z}[1/p]); \mathbb{Z}/p^\nu(j))$$

the étale cohomology groups of the scheme $\operatorname{Spec}(\mathbb{Z}[1/p])$ with coefficients in the j-th Tate twist of the p-adic integers. It is known that when $j \neq 0$ these groups vanish unless $i = 1, 2$. It was shown by Dwyer and Friedlander [17] and (independently) by Soulé [48], that when $m = 2j - i > 1$ and $i = 1, 2$, there is a surjective Chern map

$$K_m(\mathbb{Z}) \to H^i_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}_p(j)).$$

Recall the following facts about these groups:

1. When $p > j + 1$ the groups $H^1_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}_p(j))$ vanish.
2. When $j > 0$ is even, the order of $H^2_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}_p(j))$ is equal to the p-part of the numerator of B_j/j (this is a consequence of the "main Iwasawa's conjecture"[41], see [34] or [49] for more details).

Hence, those groups are not known when $i = 2$ and j is odd (assuming $p > j+1$). At the level of m it means that m is divisible by 4. Let $\mathbb{Q}(\zeta_p)$ be the cyclotomic extension of $\mathbb{Q}$ obtained by adding p-th roots of unity. Let C be the p-Sylow subgroup of the class group of $\mathbb{Q}(\zeta_p)$. The group $\Delta = \operatorname{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p)^\times$ acts upon C via the Teichmüller character

$$\omega \colon \Delta \to (\mathbb{Z}/p)^\times,$$

with $g(x) = x^{\omega(g)}$ and $x^p = 1$. For all $i \in \mathbb{Z}$ let

$$C^{(i)} = \{x \in C \quad \text{such that} \quad g(x) = \omega(g)^i x \quad \text{for all} \quad g \in \Delta\}.$$

Let C^+ be the subgroup of C fixed by the complex conjugation of $\mathbb{Q}(\zeta_p)$. The Kummer-Vandiver conjecture states that $C^+ = 0$ for arbitrary p. By the above construction, it turns out that C^+ is the direct sum of the groups $C^{(i)}$ for i even and $0 \leqslant i \leqslant p - 3$. We then deduce the reformulation of the Kummer–Vandiver conjecture [34, 49]:

Conjecture 7.1 (Kummer–Vandiver conjecture)**.** *The groups $C^{(i)}$ vanish for i even and* $0 \leqslant i \leqslant p - 3$.

From *op. cit.*, using the above setting, we get a surjective map

$$K_{2m-2}(\mathbb{Z}) \twoheadrightarrow C^{(p-m)}.$$

As consequence of (6.4), we get

Corollary 7.1. *The groups $C^{(p-5)}$ are zero for all prime $p > 3$.*

***Remark* 7.2.** By [49], we know that $C^{(p-n)}$ is zero for p "large enough" with respect to n. From computations done by Buhler and Harvey [10], we know that the conjecture is true for all (irregular) primes $p < 163577856$. This was recently improved to $p < 2147483648$ by Hart, Harvey, and Ong [28].

References

[1] D. Arlettaz. The Hurewicz homomorphism in algebraic K-theory. *J. Pure Appl. Algebra*, 71(1):1–12, 1991. 19

[2] R. Bacher. On the number of perfect lattices. *J. Théor. Nombres Bordeaux*, 30(3):917–945, 2018. 3

[3] E. S. Barnes. The complete enumeration of extreme senary forms. *Philos. Trans. Roy. Soc. London. Ser. A.*, 249:461–506, 1957. 3

[4] A.-M. Bergé and J. Martinet. On perfection relations in lattices. *Contemp. Math.*, 493:29–49, 2009. 8

[5] A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974. 18

[6] A. Borel and J.-P. Serre. Corners and arithmetic groups. *Comment. Math. Helv.*, 48:436–491, 1973. Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault. 2, 16

[7] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993). 8

[8] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. 1, 5, 16

[9] B. Brück, J. Miller, P. Patzt, R. Sroka, and J. C. H. Wilson. On the codimension-two cohomology of $\mathrm{SL}_n(\mathbb{Z})$. *Adv. Math.*, 451, 2024. 19

[10] J. P. Buhler and D. Harvey. Irregular primes to 163 million. *Math. Comp.*, 80(276):2435–2435, 2011. 20

[11] T. Church, B. Farb, and A. Putman. A stability conjecture for the unstable cohomology of $\mathrm{SL}_n(\mathbb{Z})$, mapping class groups, and $\mathrm{Aut}(F_n)$. *Contemp. Math.*, page 55–70, 2014. 17

[12] T. Church and A. Putman. The codimension-one cohomology of $\mathrm{SL}_n(\mathbb{Z})$. *Geom. Topol.*, 21(2):999–1032, Mar 2017. 18, 19

[13] H. Cohen. *A course in computational algebraic number theory*, volume 138 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 1993. 12

[14] M. Dutour Sikirić, K. Hulek, and A. Schürmann. Smoothness and singularities of the perfect form and the second Voronoi compactification of $\mathcal{A}_g$. *Algebr. Geom.*, 2(5):642–653, 2015. 13

[15] M. Dutour Sikirić, A. Schürmann, and F. Vallentin. Classification of eight-dimensional perfect forms. *Electron. Res. Announc. Amer. Math. Soc.*, 13:21–32, 2007. 3

[16] M. Dutour Sikirić and W. P. J. van Woerden. The lattice packing problem in dimension 9 by voronoi's algorithm, 2025. Preprint at `arXiv:2508.20719`. 3

[17] W. G. Dwyer and E. M. Friedlander. Algebraic and etale K-theory. *Trans. Amer. Math. Soc.*, 292(1):247–280, 1985. 20

[18] P. Elbaz-Vincent. Perfect forms of rank $\leqslant 8$, triviality of $K_8(\mathbb{Z})$ and the Kummer/Vandiver conjecture. In Renaud Coulangeon, Benedict H. Gross, and Gabriele Nebe, editors, *Lattices and Applications in Number Theory*, volume 13(1), 2016. 2

[19] P. Elbaz-Vincent, H. Gangl, and C. Soulé. Perfect forms, K-theory and the cohomology of modular groups. *Adv. Math.*, 245:587–624, 2013. 1, 4, 5, 14, 16

[20] F. T. Farrell. An extension of Tate cohomology to a class of infinite groups. *J. Pure Appl. Algebra*, 10(2):153–161, 1977/78. 16

[21] L. Fousse, G. Hanrot, V. Lefèvre, P. Pélissier, and P. Zimmermann. MPFR: a multiple-precision binary floating-point library with correct rounding. *ACM Trans. Math. Software*, 33(2):Art. 13, 15, 2007. 11

[22] K. Fukuda. cdd. https://www.inf.ethz.ch/personal/fukudak/cdd_home, 2016. 11

[23] C.-F. Gauss. Untersuchungen über die eigenschaften der positiven ternären quadratischen formen von ludwig august seeber. *J. Reine Angew. Math.*, 20:312–320, 1840. 3

[24] The GLPK group. glpk. https://www.gnu.org/software/glpk, 2017. 11

[25] The PARI group. *PARI/GP, Versions* `2.1 - 2.4`. 8, 16

[26] S. Grushevsky, K. Hulek, and O. Tommasi. Stable betti numbers of (partial) toroidal compactifications of the moduli space of abelian varieties. In *Proceedings in honour of Nigel Hitchin's 70th birthday, Volume II*, page 581–610. Oxford University Press, 2018. With an appendix by Mathieu Dutour Sikirić. 2

[27] G. Guennebaud, B. Jacob, et al. Eigen v3. http://eigen.tuxfamily.org, 2010. 11
[28] W. Hart, D. Harvey, and W. Ong. Irregular primes to two billion. *Math. Comp.*, 86(308):3031–3049, 2017. 20
[29] D.-O. Jaquet-Chiffelle. Énumération complète des classes de formes parfaites en dimension 7. *Ann. Inst. Fourier (Grenoble)*, 43(1):21–55, 1993. 3
[30] W. Keller, J. Martinet, and A. Schürmann. On classifying Minkowskian sublattices. *Math. Comp.*, 81(278):1063–1092, 2012. With an appendix by Mathieu Dutour Sikirić. 8, 9, 10, 12
[31] A. N. Korkin and E. I. Zolotarev. Sur les formes quadratiques positives quaternaires. *Math. Ann.*, 5(1):581–583, 1872. 3
[32] A. N. Korkin and E. I. Zolotarev. Sur les formes quadratiques positives. *Math. Ann.*, 11(1):242–292, 1877. 3
[33] A. Kupers. A short proof that $K_8(\mathbb{Z}) \cong 0$. https://www.utsc.utoronto.ca/people/kupers/wp-content/uploads/sites/50/2021/01/k8zshorter.pdf, 2017. 2
[34] M. Kurihara. Some remarks on conjectures about cyclotomic fields and K-groups of $\mathbb{Z}$. *Compos. Math.*, 81(2):223–236, 1992. 20
[35] J. Kuzmanovich and A. Pavlichenkov. Finite groups of matrices whose entries are integers. *Amer. Math. Monthly*, 109(2):173–186, 2002. 15
[36] J.-L. Lagrange. Démonstration d'un théorème d'arithmétique. *Nouv. Mém. Acad. Berlin, in Oeuvre de Lagrange III*, 20:189–201, 1770. 3
[37] R. Lee and R. H. Szczarba. On the homology and cohomology of congruences subgroups. *Invent. Math.*, 33:15–53, 1976. 18
[38] R. Lee and R. H. Szczarba. On the torsion in $K_4(\mathbb{Z})$ and $K_5(\mathbb{Z})$. *Duke Math. J.*, 45(1):101–129, 1978. 1, 14
[39] J. Martinet. Sur l'indice d'un sous-réseau. In *Réseaux euclidiens, designs sphériques et formes modulaires*, volume 37 of *Monogr. Enseign. Math.*, page 163–211. Enseignement Math., Geneva, 2001. With an appendix by Christian Batut. 6, 8, 9
[40] J. Martinet. *Perfect lattices in Euclidean spaces*, volume 327 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2003. 1, 2, 3, 6, 7, 9
[41] B. Mazur and A. Wiles. Class fields of abelian extensions of $\mathbb{Q}$. *Invent. Math.*, 76(2):179–330, 1984. 20
[42] J. Miller, R. Nagpal, and P. Patzt. Stability in the high-dimensional cohomology of congruence subgroups. *Compos. Math.*, 156(4):822–861, 2020. 19
[43] D. Quillen. Finite generation of the groups K_i of rings of algebraic integers. *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 179–198. Lecture Notes in Math., Vol. 341, 1973. 18
[44] D. Quillen. Higher algebraic K-theory. I. *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341, 1973. 17
[45] A. Schrijver. *Theory of linear and integer programming*. Wiley-Interscience Series in Discrete Mathematics. John Wiley & Sons, Ltd., Chichester, 1986. A Wiley-Interscience Publication. 10
[46] A. Schürmann. Enumerating perfect forms. In *Quadratic forms—algebra, arithmetic, and geometry*, volume 493 of *Contemp. Math.*, page 359–377. Amer. Math. Soc., Providence, RI, 2009. 3
[47] C. Soulé. The cohomology of $SL_3(\mathbb{Z})$. *Topology*, 17(1):1–22, 1978. 1
[48] C. Soulé. K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale. *Invent. Math.*, 55:251–296, 1979. 20
[49] C. Soulé. Perfect forms and the Vandiver conjecture. *J. Reine Angew. Math.*, 517:209–221, 1999. 20
[50] C. Soulé. On the 3-torsion in $K_4(\mathbb{Z})$. *Topology*, 39(2):259–265, 2000. 5
[51] W. P. J. van Woerden. An upper bound on the number of perfect quadratic forms. *Adv. Math.*, 365:107031, 2020. 3
[52] G. Voronoi. Nouvelles applications des paramètres continus à la théorie des formes quadratiques 1: Sur quelques propriétés des formes quadratiques positives parfaites. *J. Reine Angew. Math*, 133(1):97–178, 1908. 1, 2, 3, 4
[53] C. A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K-theory. 19, 20

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