“READ ME” FOR THE NUMERICAL DATA OF GRAMINDEX.GP

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Abstract. This note is a short user’s guide to the `PARI-GP`-file `Gramindex.gp`. It contains Gram matrices for lattices of dimension $n \leq 9$, for each of the existing lattice types according to the classification in [KMS10].

1. According to the notation used in [KMS10], Λ is an n-dimensional lattice, Λ' is a sublattice of Λ equipped with a basis $\mathcal{B}_0 = (e_1, ..., e_n)$ of minimal vectors of Λ,
$\imath$ is the index $[\Lambda : \Lambda']$,
d is the annihilator of Λ/Λ'.

2. Annihilators $d > 1$ up to $n = 9$:

$$d = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12\,;$$

Indices > 1 up to $n = 9$; cyclic: see list of annihilators; non-cyclic:

$$2^2, 4 \cdot 2, 2^3, 3^2, 6 \cdot 2, 4 \cdot 2^2, 2^4\,,$$

with the usual convention for finite Abelian groups.

3. With each code C over $\mathbb{Z}/d\mathbb{Z}$ we associate in [Mar01] and [KMS10] a well-defined *minimal class* $\mathcal{C}_C$, characterized by the fact that the kissing number s of the class is minimal among all classes which admit the code C; we display only *one* Gram matrix per code.

4. The file `Gramindex.gp` can be read in a PARI-GP session; each matrix of the file has a name, the choice of which we explain below. The file can also be edited (under any editor, e.g., emacs, vi,...); one can then read the name of each matrix and also the code, on a line beginning with a “\\” mark.

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5. A matrix name has the shape

$$\mathrm{a}n\mathrm{x}n\text{'},$$

where $n \in \{4,5,6,7,8,9\}$ is the dimension and the letter x is one of a,b,c,..., and n' indicates the structure of the quotient Λ/Λ': $2,\dots,10$ and 12 in the cyclic case, otherwise 22 for 2^2, 42 for $4\cdot 2,\dots$, 422 for $4\cdot 2^2$, and 2222 for 2^4; thus when $n' > 12$, $\imath(\Lambda)$ is the product of the figures which occur in n'. The choice for x depends on the specific code corresponding to a quotient of type n'.

6a. (General form.) The codewords are written in the form $(a_1,\dots,a_n)$ (or $(a_1,\dots,a_n)_d$ if we wish to recall the annihilator) with $\frac{0}{<}a_i \le \frac{d}{2}$. *It is assumed that for each subscript $j \in [1,n]$, at least one codeword has a non zero component at j.* (Otherwise, the pair (Λ,Λ') would trivially come from a lower dimension.)
6b. (Cyclic case). Set $d' = \lfloor \frac{d}{2} \rfloor$. Negating some e_i if needed, we may moreover assume that $1 \le a_1 \le \cdots \le a_n \le d'$. We then denote (as in [KMS10]) by m_i the number of coefficients a_j which are equal to i, and write down the code in the form $(m_1, m_2, \dots, m_{d'})$ where the m_i are non-negative and add to n; for instance,

$$(3,4,2)_6 \quad \text{reads} \quad (1,1,1,2,2,2,2,3,3)\,.$$

6c. (Averaging in the cyclic case.) The Gram matrices are then constructed using equal scalar products $e_k \cdot e_\ell = x_i$ for k,ℓ relative to a same m_i and $e_k \cdot e_\ell = y_{i,j}$ for k relative to m_i and $\ell > k$ relative to m_j; moreover, if d is even, $x_{d'}$ and the $y_{i,d'}$ are zero. We need this way at most $\frac{n'(n'+1)}{2}$ parameters, and at most $\frac{n'(n'-1)}{2}$ if d' is even. Note that there is no x_i if $m_i = 0$ or 1, and no $y_{i,j}$ if m_i or m_j is zero.

7. (Equivalent codes.) The tables in `Gramindex.gp` show one code in every equivalence class of codes. The way this can be performed is standard for binary codes (i.e., codes with $d = 2$). For cyclic codes over $\mathbb{Z}/d\mathbb{Z}$, classes are obtained using the transformation $x \mapsto kx \mod d$ with $(k,d) = 1$, and re-ordering the new $|a_i| \mod d$, which induces an action of $(\mathbb{Z}/d\mathbb{Z})^\times/\pm 1$. This action is obtained by performing one of the three transformations below:
• A transposition, namely (m_1,m_2) if $d = 5$, (m_1,m_3) if $d = 8$, (m_1,m_5) if $d = 12$;
• A cycle of order 3, namely (m_1,m_2,m_3) if $d = 7$, (m_1,m_2,m_4) if $d = 9$;
• The double transposition $(m_1,m_3)\cdot(m_2,m_4)$ if $d = 10$.

8. (Construction of the Gram matrices.) We start with the Gram matrix M_0 of the basis $\mathcal{B}_0 = (e_1, \dots, e_n)$ for Λ', and replace in $\mathcal{B}_0$ some e_i by convenient vectors of Λ.

a. (Cyclic case.) We have $\Lambda = \langle \Lambda', e \rangle$ with e of the form

$$\frac{a_1 e_1 + \cdots + a_n e_n}{d},$$

and the first m_1 coefficients equal to 1, the next m_2 equal to 2, etc. Within the range of our table, m_1 is never zero, so that $(e, e_2, \dots, e_n)$ is a basis for Λ. The displayed Gram matrix for Λ is then that of this basis. Its diagonal entries except the first one are equal to the minimum of Λ, and we have $e_1 = d(e - \sum_{i \ge 2} a_i e_i)$.

b. (Quotient of rank 2.) In this case we may write $\Lambda = \langle \Lambda', e, f \rangle$ with e, f defined by the corresponding code. In case the code is not symmetric, which occurs when Λ/Λ' is of type $4 \cdot 2$ or $6 \cdot 2$, e is relative to denominator $d = 4$ or 6 and f to denominator 2. We have chosen e and f in such a way that $\mathcal{B} = (e, e_2, \dots, e_{n-1}, f)$ is a basis for Λ. The displayed Gram matrix for Λ is then that of this basis.

c. (Quotients of type 2^3, $4 \cdot 2^2$ and 2^4.) Here the construction of the Gram matrix is explained in each case. As above, for the quotient of type $4 \cdot 2^2$, e has denominator 4.

9. (Normalization). In dealing with the index theory in [KMS10], we used to scale all lattices to minimum 1. However, for practical reasons, the Gram matrices displayed in the file `Gramindex.gp` *are scaled to the smallest minimum which make them integral*. For instance, matrices with $d = 2$ will in general have minimum $\mu = 4$, except in the case of a doubly even code, where $\mu = 2$.

References

[Mar01] J. Martinet, *Sur l'indice d'un sous-réseau*, in "Réseaux euclidiens, designs sphériques et formes modulaires", J. Martinet ed., Monographie Ens. Math. **37**, Genève (2001), 165–213.

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