Bases of minimal vectors in lattices, II

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Abstract. We prove that a Euclidean lattice of dimension $n = 5$ (resp. 6; resp. 7) having at least 6 (resp. 10; resp. 18) pairs of minimal vectors has a basis of minimal vectors.

Résumé. Nous montrons qu'un réseau euclidien de dimension $n = 5$ (resp. 6; resp. 7) ayant au moins 6 (resp. 10; resp. 18) paires de vecteurs minimaux possède une base de vecteurs minimaux.
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1. Introduction

In this paper, we consider *well rounded lattices*, i.e. lattices Λ in some Euclidean space E whose set of minimal vectors spans E. We denote by $S(\Lambda) = S$ the set of minimal vectors of Λ and by $2s$ its cardinality. Let $n = \dim E$. Our aim is to calculate lower bounds s_0 for s which ensure that any well rounded lattice with $s \geq s_0$ is generated by its minimal vectors. It is proved in [5] that up to $n = 8$, every lattice generated by its minimal vectors possesses a basis of minimal vectors. Hence for the dimensions we shall consider here, these two notions are equivalent. We shall prove:

Theorem 1.1. *Let Λ be an n-dimensional, well rounded lattice. Suppose that one of the following conditions holds:*

1. $n \leq 4$;
2. $n = 5$ *and* $s \geq 6$;
3. $n = 6$ *and* $s \geq 10$;
4. $n = 7$ *and* $s \geq 18$.

Then Λ possesses a basis of minimal vectors. Moreover, for $n = 5$, 6 and 7, the lower bounds given above for s are optimal.

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Since perfect lattices satisfy the condition $s \geq \frac{n(n+1)}{2}$, we recover the result proved by Csóka ([1], 1987), namely that perfect lattices of dimension $n \leq 7$ possess bases of minimal vectors, a result which we can check nowadays on the Stacey-Jaquet classification.

Given a well rounded lattice Λ, n independent minimal vectors $e_1, \dots, e_n$ of Λ constitute a basis for a sublattice Λ' of Λ. The set of possible structures for Λ/Λ', and in particular the list $\mathcal{L}$ of indices $[\Lambda : \Lambda']$ are invariants for Λ.

Definition 1.2. *The largest possible value $\imath(\Lambda)$ for $[\Lambda : \Lambda']$ is the* maximal index of Λ*. For a given pair (Λ, Λ'), the smallest integer m such that $\Lambda = \langle \Lambda', e^{(i)} \rangle$ where Λ' has a basis $(e_1, \dots, e_n)$ with $e_i \in S(\Lambda)$ and the $e^{(i)}$ are of the form $\frac{a_1 e_1 + \dots + a_m e_m}{d_i}$ is called the* length of (Λ, Λ').

For instance, if $\imath(\Lambda) = 2$ (resp. 3), the length of Λ is the smallest integer m such that one can write $\Lambda = \langle \Lambda', e \rangle$ with $e = \frac{e_1 + \dots + e_m}{d}$ and $d = 2$ (resp. $d = 3$). Note that Λ has a basis of minimal vectors if and only if index 1 occurs in the list $\mathcal{L}$.

Lemma 1.3. (Watson's index lemma, [6]) *Suppose that $\Lambda = \langle \Lambda', e \rangle$ for a vector $e = \dfrac{a_1 e_1 + \dots + a_m e_m}{d}$ with $a_i \geq 1$ and $d \geq 2$. Then*

$$\sum_{i=1}^{n} a_i \big(N(e - e_i) - N(e_i) \big) = \Big(\Big(\sum_{i=1}^{n} a_i \Big) - 2d \Big) N(e)\,.$$

Assume moreover that the vectors e_i are minimal. Then we have $\sum_i a_i \geq 2d$ and equality holds if and only if all vectors $e - e_i$ are minimal.

Proof. The straightforward proof is left to the reader. Some more details can be read in [4], Section 2. ☐

The following proposition is an example of the results we shall prove in relation with the index:

Proposition 1.4. *Let Λ be a well rounded lattice of dimension n, maximal index 2, and length ℓ, having no basis of minimal vectors.*

1. *If $\ell = n$, then $s(\Lambda) = n$.*
2. *If $\ell = n - 1$, then $s \leq 2n - 1$, and if $s \geq n + 4$, we can choose the e_i so that either $S(\Lambda) = \{\pm e_i, \pm(e_n + e_j)\}$, $i = 1, \dots, n$, $j = 2, \dots, s - n$, or $S(\Lambda) = \{\pm e_i, \pm(e_1 + e_j + e_n)\}$, $i = 1, \dots, n$, $j = 1, \dots, s - n$ (and then, $s \leq 2n - 2$).*

Numerical evidence suggests that the bound $s \leq 2n - 1$ is optimal from dimension 7 onwards. Theorem 1.1 in dimension 6 will follow from the improvement $s \leq 2n - 3 = 9$.

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2. Bounds for the Index

We succinctly recall some results on the index. For the proof, the reader is referred to [4].

The trivial bound $\imath \le \gamma_n^{n/2}$ is optimal for all $n \le 8$, but if we disregard the root lattices $\mathbb{D}_4$, $\mathbb{D}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ (which obviously have bases of minimal vectors), we obtain the bounds $\imath \le 2, 3, 4, 8$ for $n = 5, 6, 7, 8$, and we moreover have $\imath = 1$ for $n \le 4$, which proves Theorem 1.1 for dimensions $n \le 4$.

When $d = 2$ or 3, we may suppose (by reduction modulo d of the a_i and negation of some e_i) that we have $a_1 = \cdots = a_m = 1$ and $a_i = 0$ for $m \le i \le n$ for some $m \le n$. The index lemma then shows that if Λ is not generated by its minimal vectors, and if $\imath(\Lambda) = 2$ (resp. 3), then Λ has length $\ell \ge 5$ (resp. $\ell \ge 7$).

We now make and comment two general remarks, namely:

1. if Λ/Λ' is cyclic of prime order and if Λ is not generated by minimal vectors, then $S(\Lambda)$ must be reduced to $S(\Lambda')$;
2. if $[\Lambda : \Lambda'] = \imath(\Lambda)$, then $\imath(\Lambda') = 1$.

Condition $\imath(\Lambda') = 1$ can be expressed in terms of *characteristic determinants* (in the sense of Korkine and Zolotareff; see [3], Section 5.1). The consideration of determinants of orders 1, 2, 3 show that a lattice of maximal index $\imath = 1$ satisfies the properties below.

1. Minimal vectors have components $0, \pm 1$ on $e_1, \dots, e_n$.
2. Two minimal vectors cannot have components $(1, 1)$ and $(1, -1)$ on a pair (i, j) of indices.
3. Three minimal vectors cannot have components $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ on a system (i, j, k) of three components.

The easy proposition below illustrates the case of these small indices:

Proposition 2.1. *Let Λ be a well rounded lattice which is not generated by its minimal vectors, of dimension and length n, and of index 2 (resp. 3). Then $s(\Lambda) = n$ (resp. $s(\Lambda) \le n + \lfloor \frac{n}{2} \rfloor$).*

Proof. We may write $\Lambda = \langle \Lambda', e \rangle$ with

$$e = \frac{a_1 e_1 + \cdots + a_n e_n}{d} \quad \text{and} \quad d = 2 \text{ or } 3\,.$$

Let $x = \sum_{i=1}^n x_i e_i \in S(\Lambda')$; by the remarks above, we have $x_i = 0, \pm 1$.

Suppose first that $d = 2$. Negating e_i and replacing e by $e - e_i$ if necessary, we may assume that all x_i are equal to 0 or 1. If, say, $x = e_1 + \cdots + e_k$ were minimal for some $k \ge 2$, we could write $e = \frac{x + e_{k+1} + \cdots + e_n}{2}$ and Λ would have length $\ell \le n + 1 - k$, a contradiction.

Suppose next that $d = 3$. We get rid of sums $e_1 + \cdots + e_k$ with $k \ge 2$ as above, but we must this time also consider sums of the form $e_1 \pm e_2 \pm \cdots \pm e_k$. If, say, $x = e_1 + e_2 - e_3$ were minimal, we could write $e - e_3 = \frac{x - e_3 + e_4 + \cdots + e_n}{3}$, and Λ would have length $\ell \le n - 1$. Similarly $x = e_1 - e_2$ and $y = e_1 - e_3$ cannot be both minimal, for we could then write $e-e_1=\frac{-x-y+e_4+\cdots+e_n}{3}$. Then the minimal vectors of Λ' others that the $\pm e_i$ are vectors $\pm(e_i-e_j)$ with disjoint supports $\{i,j\}$. This completes the proof of Proposition 2.1. □

This proposition immediately implies Theorem 1.1 for dimension 5, since 5-dimensional lattices have maximal index $\imath=1$ or 2.

3. Lattices of Index 2

In this section, we prove Proposition 1.4 and complete the proof of Theorem 1.1 for dimension 6. We also prove a preliminary result for dimension 7.

Proof of Proposition 1.4. The first assertion of the proposition results from Proposition 2.1. Let us prove (2). We then write as usual

$$\Lambda=\langle\Lambda',e\rangle \ \text{ with } \ e=\frac{e_1+\cdots+e_{n-1}}{2}\,.$$

As in the proof of Proposition 2.1, we show that $S(\Lambda)=S(\Lambda')$ and that at least one $x\in S(\Lambda')$ can be written as a sum $e_{i_1}+\cdots+e_{i_k}$. Since systems of components $(1,1),(1,-1)$ are forbidden, every $x\in S(\Lambda')$ (up to sign) has then positive components. Moreover, by definition of the length, x must have a non-zero component on e_n. If, say, $x=e_1+e_2+e_3+e_n$ were minimal, we could write $e=\frac{x-e_n+e_4+\cdots+e_{n-1}}{2}$. As a consequence, we may assume that minimal vectors other than the e_i are of the form $\pm(e_i+e_n)$ (type I) or $\pm(e_i+e_j+e_n)$ (type II). Next we observe that $x=e_1+e_2+e_n$ and $y=e_3+e_n$ cannot be both minimal, for we could write $e+e_n=\frac{x+y+e_4+\cdots+e_{n-1}}{2}$, nor similarly x and $y=e_3+e_4+e_n$. We now discuss the various possibilities according to the number t of type I vectors.

• $t\ge 2$, say, e_1+e_n and $e_2+e_n\in S(\Lambda')$. The only possible vector of type II is $e_1+e_2+e_n$, and then $s=n+3$. This proves Proposition 1.4 in this case.

• $t=1$, say, $e_1+e_n\in S(\Lambda')$. Then all type II vectors must be of the form $e_1+e_i+e_n$. Replacing e_n by $e'_n=e_1+e_n$ and e_1 by $e'_1=-e_1$, we reduce ourselves to the previous case.

• $t=0$. We cannot have a system $e_i+e_j+e_n$, $e_i+e_k+e_n$, $e_j+e_k+e_n$, which would imply $i(\Lambda')\ge 2$ (the vectors above define a non-trivial characteristic determinant). Hence we may assume (up to a permutation of $e_1,\dots,e_{n-1}$) that all minimal vectors other than the $\pm e_i$ are of the form $e_1+e_i+e_n$. □

Proof of Theorem 1.1 *for dimension* 6. We consider a lattice Λ with maximal index $\imath$. We have $\imath\le 4$, and bases of minimal vectors exist if $\imath=4$ (because Λ is then similar to $\mathbb{D}_6$; see [4], theorem 4.3), if $\imath=3$ (because equality holds in Watson's theorem) and of course if $\imath=1$, and also if $\imath=2$ and if Λ has length 4 (because it then has a $\mathbb{D}_4$-section) or 6 (by proposition 1.4).

There remains to consider the case where Λ is of the form $\langle\Lambda',e\rangle$ with $e=\frac{e_1+e_2+e_3+e_4+e_5}{2}$, that we scale for convenience to minimum 2. Hence, $e_i\cdot e_i=2$. By Proposition 2.1, we may assume that up to sign, the minimal vectors of Λ' other than the $\pm e_i$ are the vectors e_i+e_6 or $e_i+e_5+e_6$ for $i=1,\dots,k$, $k\le 5$ or $k\le 4$, and we must prove that $k\le 3$.

Suppose first that $e_1+e_6,\dots,e_4+e_6$ are minimal. Applying the averaging argument of [4], Proposition 8.5 to the group S_4 acting on $\{1,2,3,4\}$, we may reduce ourselves to the case where the scalar products $e_i\cdot e_j$ only depend on three parameters, namely $t=e_i\cdot e_j$ $(1\le i<j\le 4)$, $u=e_i\cdot e_5$ $(1\le i\le 4)$ and $v=e_5\cdot e_6$ (we have $e_i\cdot e_6=-1$ for $1\le i\le 4$). An easy calculation shows that

(1) $N(e+e_6)-2=-\frac{3}{2}+3t+2u+v$;

(2) $N(e-e_5+e_6)-2=-\frac{3}{2}+3t-2u-v$;

(3) $N(e-e_2-e_3)-2=\frac{1}{2}-t$.

The left hand sides must be strictly positive. Adding (1) and (2), we obtain the lower bound $6t>3$ which contradicts (3).

Suppose next that $e_1+e_5+e_6,\dots,e_4+e_5+e_6$ are minimal. Using the same averaging argument, we find this time $N(e-e_1)-2=u+\frac{1}{2}$ and $N(e-e_1+e_6)-2=v-u-\frac{3}{2}$. Adding these two positive numbers, we obtain the inequality $-v-1>0$, i.e. $v<-1$. This contradicts $N(e_5+e_6)=4+2v\ge 2$.

That $2s\le 18$ is the least upper bound for the number of minimal vectors in a well rounded 6-dimensional lattice can be seen by inspecting the Gram matrices displayed in the appendix. □

We now turn to lattices of index 2 and length $n-2$ not generated by their minimal vectors. This implies $n\ge 7$ by the index lemma. We put special emphasis on dimension $n=7$, for which *hyperplane* sections can be handled using the case $n=6$ of Theorem 1.1, the statement of which is more precise than that of Proposition 1.4. This will complete the study of 7-dimensional lattices of index 2. The bounds given in the following proposition are perhaps not optimal.

Proposition 3.1. *Let Λ be a well rounded lattice of dimension n, index 2, and length $n-2$, having no basis of minimal vectors. Then $s\le 4n-5$, and if $n=7$, we have the sharper bound $s\le 17$.*

Proof. Let F be the span of $e_1,\dots,e_{n-2}$ and let $F_1=F+e_{n-1}$, $F_2=F+e_n$, $F_3=F+(e_{n-1}+e_n)$ and $F_4=F+(e_{n-1}-e_n)$. Since Λ' has index 1, all minimal vectors of Λ' are of the form $\pm e_i\pm e_j\pm\dots$, hence (up to sign) belong to one of the affine spaces F_i. Moreover, F_3 and F_4 cannot both contain minimal vectors of Λ', because their components on e_{n-1} and e_n would define a characteristic determinant equal to ± 2; see Section 2. Negating e_n if necessary, we may assume that $S(\Lambda')\cap F_4=\emptyset$.

By Proposition 1.4 applied in dimension $n-1$, there are in $F+\mathbb{R}e_{n-1}$ and $F+\mathbb{R}e_n$ at most $n-2$ pairs of minimal vectors besides $e_1,\dots,e_{n-1},e_n$. Hence each set F_1, F_2 contributes at most $n-2$ to s. A result of the same kind holds for F_3: if $e_{n-1}+e_n$ is minimal, the contributions to s of F_3 is now at most $n-1$ (at most $(n-2)$, plus 1 for $e_{n-1}+e_n$); otherwise, if $S(\Lambda')\cap F_3\ne\emptyset$, let x be one of its elements, and let Λ'' be the lattice generated by $e'_1=e_1,\dots,e'_{n-1}=e_{n-1}$ and $e'_n=x$; applying the previous argument to $F'=F+e'_n$, we find that there are at most $n-2$

minimal vectors in F' besides $e_1, \dots, e_{n-1}, x$, but this list includes e_n. Altogether, this gives for $s(\Lambda) = s(\Lambda')$ the upper bound $s \le n + 2(n-2) + (n-1) = 4n - 5$, which proves the first part of the proposition.

The argument for $n = 7$ is similar, replacing the bound $n-2$ of Proposition 1.4 by the bound $3 = 9 - 6$ of Theorem 1.1 for dimension 6. This time we have $s \le 7 + 2 \times 3 + 4 = 17$. $\square$

4. 7-Dimensional Lattices

We begin this section with a lemma which paves the way for the complete study of dimension 7. We keep the previous notation. In particular, $e_1, \dots, e_7$ are independent minimal vectors of Λ which constitute a basis for a sublattice Λ' of Λ.

Lemma 4.1. *Let Λ be a 7-dimensional lattice which is not generated by its minimal vectors. Assume that $s(\Lambda) \ge 14$. Then Λ has one of the following forms:*

1. $\Lambda = \langle \Lambda', e \rangle$ *with* $e = \dfrac{e_1 + e_2 + e_3 + e_4 + e_5}{2}$.
2. $\Lambda = \langle \Lambda', e, f \rangle$ *with*
   $e = \dfrac{e_1 + e_2 + e_3 + e_4}{2}$ *and* $f = \dfrac{e_1 + e_2 + e_5 + e_6 + e_7}{2}$.
3. $\Lambda = \langle \Lambda', e \rangle$ *with* $e = \dfrac{e_1 + e_2 + e_3 + e_4 + e_5 + 2e_6 + 2e_7}{4}$.

Proof. For $n = 7$, we have $\imath(\Lambda) \in \{2, 3, 4\}$ or $\imath = 8$. but this last case must be discarded since it occurs only for lattices similar to $\mathbb{E}_7$, see the beginning of Section 2. If $\imath(\Lambda) = 2$ and $\mathrm{length}(\Lambda) \ge 6$, we have $s \le 13$ by Proposition 1.4. Hence if $\imath = 2$, we must have $\mathrm{length}(\Lambda) = 5$, which is case one of the lemma. If $\imath(\Lambda) = 3$, we have $s \le 10$ by Proposition 2.1. We are thus left with lattices such that $[\Lambda : \Lambda'] = 4$.

If Λ/Λ' is non-cyclic, we attach to (Λ, Λ') the binary code (of length 7 and dimension 2), with words $(a_1, \dots, a_7) \mod 2$ for $\frac{a_1 e_1 + \cdots + a_7 e_7}{2} \in \Lambda$. Its minimal weight is ≥ 4 by Watson's inequality, which implies that its weight system is (4^3), $(4^2, 6)$ or $(4, 5^2)$. In the first two cases, Λ has a basis of minimal vectors, whereas the third case corresponds to case (2) of the lemma.

If Λ/Λ' is cyclic, we can write $\Lambda = \langle \Lambda', e \rangle$ with $e = \frac{e_1 + \dots e_p + 2e_{p+1} + \cdots + 2e_7}{4}$ and $4 \le p \le 6$ (see [4], Proposition 5.1). Applying Watson's identity if $p = 6$ or writing $e = \frac{e' + e_5 + e_6 + e_7}{2}$ with $e' = \frac{e_1 + e_2 + e_3 + e_4}{2}$ if $p = 4$, we see that $e - e_7$ is in both cases a minimal vector. Hence we must have $p = 5$, which is case (3) of the lemma. $\square$

Remark 4.2. The smallest possible value for s is $s = 7$ in cases (1) and (3), and $s = 15$ in case (2); see [4], table 11.1.

We now consider lattices of index $\imath \ge 4$ which are not generated by their minimal vectors. This implies $\imath = 4$, and we are in one of the cases (2), (3) of Lemma 4.1. We keep the notation of the previous sections. We begin with case (2), corresponding to a non-cyclic quotient Λ/Λ'.

Proposition 4.3. *Let $e_1, \dots, e_7$ be 7 independent minimal vectors of Λ generating a sublattice Λ' of Λ. Suppose that Λ/Λ' is non-cyclic of order 4 and that Λ is not generated by its minimal vectors. Then $s(\Lambda) \le 15$.*
(The upper bound $s(\Lambda) \le 15$ is optimal by Remark 4.2.)

Proof. We have $\imath(\Lambda) = 4$ and by lemma 4.1, we may write $\Lambda = \langle \Lambda, e, f\rangle$ with $e = \dfrac{e_1+e_2+e_3+e_4}{2}$ and $f = \dfrac{e_1+e_2+e_5+e_6+e_7}{2}$. We observe that Λ contains at least 15 pairs of minimal vectors, namely the 7 pairs $\pm e_i$ and the 8 pairs $\frac{\pm e_1 \pm e_2 \pm e_3 \pm e_4}{2}$. We are going to show that Λ has no other minimal vectors.

Set $f' = \dfrac{e_3+e_4+e_5+e_6+e_7}{2}$. Since $\Lambda = \langle \Lambda', e, f\rangle = \langle \Lambda', e, f'\rangle$ is not generated by its minimal vectors, none of the cosets $f+\Lambda'$, $f'+\Lambda'$ contains minimal vectors. We first prove that the only minimal vectors in the coset Λ' are the $\pm e_i$.

Indeed, the arguments used to prove Proposition 1.4 show that we may reduce ourselves to the case where there exists a minimal vector of the form $x = e_{i_1} + \dots + e_{i_r}$ with $r \ge 2$, and that the support of x must not be contained in any of the sets $\{1,2,3,4\}$, $\{1,2,5,6,7\}$ or $\{3,4,5,6,7\}$. This implies that the support $\mathcal{T}$ of x intersects all three complementary sets $\{1,2\}$, $\{3,4\}$ and $\{5,6,7\}$ of the supports of f', f and e. In particular , we have $r \ge 3$. We cannot have $\mathcal{T} \subset \{1,2,3,4\}$ because $\frac{e_1+e_2+e_3+e_4}{2} \notin 2\Lambda$. Hence we may assume that either $x = e_1+e_3+e_5+\dots$ with $r \ge 3$ components or $x = e_1+e_2+e_3+e_5+\dots$ with $r \ge 4$ components. The case $r = 4$ (and *a fortiori* $r > 4$) is impossible: otherwise $f = \frac{x+e_2-e_3+e_7}{2}$ or $f = \frac{x-e_3+e_6+e_7}{2}$ would be minimal. Suppose now $r = 3$. The existence of the 8 pairs of minimal vectors $\frac{\pm e_1 \pm e_2 \pm e_3 \pm e_4}{2}$ implies $e_i \cdot e_j = 0$ and a straightforward computation yields $N(e_1+e_5) + N(e_3+e_5) = N(e_5) + N(e_1+e_3+e_5)$ which shows the minimality of e_1+e_5 and e_3+e_5 and contradicts the inequality $r \ge 3$ established above.

We now prove that $\frac{\pm e_1 \pm e_2 \pm e_3 \pm e_4}{2}$ are the only minimal vectors in the coset $e+\Lambda$. Let $x \in S(\Lambda) \cap (e+\Lambda)$, say, $x = \{\frac{a_1e_1+\dots+a_7e_7}{2}\}$; we have $a_i \equiv 1 \mod 2$ if $i \le 4$ and $a_i \equiv 0 \mod 2$ if $i > 4$. For an i with $a_i \ne 0$, consider the lattice Λ'' generated by x and the e_j with $j \ne i$. We have $\pm e_i = \dfrac{-2x + \sum_{j\ne i} a_j e_j}{|a_i|}$, whence $|a_i| = [\Lambda : \Lambda''] \le 4$, and equality must be excluded, since it corresponds to the case "$p = 4$" in the proof of Lemma 4.1, (3). Hence $a_i = 0, \pm 2$ for $i = 5,6,7$, say, $a_i = 2a_i'$, and at least one of them is non-zero, so that we may assume that $a_7' = 1$. Replacing e_7 in f by its expression as a combination of x and the a_j with $j \ne 5$, we obtain for f an expression of the form

$$f = \frac{\sum_{1\le i\le 4} b_i e_i + \sum_{5\le i\le 7} c_i e_i}{4}$$

with odd b_i ($b_i = 2 - a_i$ or $-a_i$) and even c_i ($2(1-a_5')$, $2(1-a_6')$, $2x$). The impossible case "$p = 4$" again shows up. $\square$

Proposition 4.4. *Let $e_1, \dots, e_7$ be 7 independent minimal vectors of Λ generating a sublattice Λ' of Λ. Suppose that Λ/Λ' is cyclic of order 4 and that Λ is not generated by its minimal vectors. Then $s(\Lambda) \le 17$.*

Proof. By Lemma 4.1, we may write Λ in the form $\Lambda = \langle \Lambda', e\rangle$ where $e = \frac{e_1+\cdots+e_5+2(e_6+e_7)}{4}$. Set $f = \frac{e_1+e_2+e_3+e_4+e_5}{2}$. Then $e = \frac{f+e_6+e_7}{2}$, and the cosets of Λ modulo Λ' are those of $\pm e$, f and 0. We again have $\imath(\Lambda) = 4$. Assuming that Λ is not generated by its minimal vectors, we are going to show that Λ has at most 17 pairs of minimal vectors. We observe that the cosets of $\pm e$ do not contain any minimal vector, and we consider successively Λ' and $f + \Lambda'$, and denote by s' and s_f the contributions to s of Λ' and $f + \Lambda'$ respectively.

Let $x \in S(\Lambda')$, $x \ne \pm e_i$. Because $\imath(\Lambda) = 4$, x is up to sign a sum $e_i \pm e_j \pm \dots$ with $i < j < \dots$. As in the proof of Proposition 4.3, the signs of e_6, e_7 do not matter, but those of $e_1, \dots, e_5$ do. If $x = e_i + e_j \pm \dots$ with $i, j \le 5$, or if $x = \cdots + e_6 + e_7$, then Λ has length $\ell \le 6$. If $x = e_i + e_j$ with $i \le 5 < j$ or $x = e_i - e_j$ with $i, j < 5$, then we are in the case $p = 6$ of Lemma 4.1. Finally, up to permutations of $\{1, \dots, 5\}$ and $\{6, 7\}$, we may assume that $x = e_1 - e_2 + e_6$. Now another vector $y = e_i - e_j + e_6$ cannot be minimal: indeed, if $y = e_1 - e_3 + e_6$ (resp. $e_3 - e_4 + e_6$) were minimal, we would have $e + e_6 - e_1 = \frac{-x-y-e_1+e_4+e_5+2e_7}{4}$ (resp. $e - e_6 - e_1 = \frac{x+y+e_5+2(e_2+e_4+e_7)}{4}$), and Λ would have length 6. Hence we have $s' \le 9$ and if equality holds, the pairs of minimal vectors in Λ' other than the $\pm e_i$ are one pair $\pm(e_i - e_j \pm e_6)$ and one pair $\pm(e_{i'} - e_{j'} \pm e_7)$.

We now turn to the coset $f + \Lambda$, whose elements are of the form $x = \frac{a_1e_1+\cdots+a_5e_5+a_6e_6+a_7e_7}{2}$ with $a_i \equiv 1 \mod 2$ for $1 \le i \le 5$ and $a_6 \equiv a_7 \equiv 0 \mod 2$. If $a_i \ne 0$, we may write $\pm e_i = \frac{2x - \sum_{j\ne i} a_j e_j}{|a_i|}$. Assume that x is minimal, and let Λ'' be the lattice generated by x and the e_j with $j \ne i$. We have $[\langle \Lambda', f\rangle : \Lambda''] = |a_i|$ hence $[\Lambda : \Lambda''] = 2|a_i|$, which implies $2|a_i| \le \imath(\Lambda) = 4$, i.e. $a_i \in \{0, \pm1, \pm2\}$. Negating some vectors among $\{e, f, e_6, e_7\}$ if necessary, we may assume that there are 0, 1 or 2 coefficients a_i equal to -1 with $i \le 5$ and that $a_6, a_7 \in \{0, 2\}$. Up to sign, we have $x = f' + b_6e_6 + b_7e_7$, where $f' = f$, $f - e_i$ or $f - e_i - e_j$ and $b_i = 0$ or 1. If $f' = f + \dots$ then $e = \frac{f'+e_6+e_7}{2}$ has length $\ell \le 3$ with respect to $\langle \Lambda', f\rangle$, which is plainly impossible. Similarly, if $f' = f - e_i + \dots$, then $e = \frac{f'+e_i+\dots}{2}$ has length $\ell \le 4$ with respect to $\langle \Lambda', f\rangle$, which is again impossible. Finally, the only possibility is $f' = f - e_i - e_j$ (with $b_6 = b_7 = 0$), which implies the upper bound $s_f \le 10$.

Taking into account s' and s_f, we obtain the bound $s \le 19$, that we shall now slightly sharpen.

We observe that two vectors $f' = f - e_i - e_j$ and $x = e_i - e_j + e_k$, $i < j \le 5$ and $k = 6, 7$ cannot be both minimal. Indeed, with $i = 1$, $j = 2$ and $k = 6$, we may write $e - e_2 = \frac{f'+x+e_7}{2}$. Hence if $s_f = 10$ or $s_f \le 8$, we have $s \le 17$. There remains to consider the case where all vectors $f' = f - e_i - e_j$ except, say, $f - e_4 - e_5$ and also $x = e_4 - e_5 + e_6$ and $y = e_4 - e_5 + e_7$ are minimal. We choose the scale in which $N(e_i) = 4$. Writing that the nine vectors $f - e_i - e_j$ $(1 \le i < j \le 5,\ (i, j) \ne (4, 5))$

are minimal, we express the Gram matrix for $\langle e_1, \dots, e_5, f\rangle$ as a function of one parameter t such that $e_i \cdot e_j = t$ for $1 \le i < j \le 3$ and $e_i \cdot e_j = 2 - t$ for $1 \le i \le 3$ and $j = 4, 5$ and for $i = 4$ and $j = 5$. We then apply the averaging argument of Proposition 8.5 in [4] referred to above with respect to the group generated by the permutations of $\{1, 2, 3\}$, the transposition $(6, 7)$, and the transformation which exchanges e_4 and $-e_5$ and negates e_6 and e_7. We can express a Gram matrix as a function of three parameters: t, u such that $e_4 \cdot e_6 = e_4 \cdot e_7 = u$ and $e_5 \cdot e_6 = e_5 \cdot e_7 = -u$, and $v = e_6 \cdot e_7$, and we have $e_i \cdot e_j = 0$ for $i = 1, 2, 3$ and $j = 6, 7$. Then $N(e_4 - e_5 + e_6) = 8 + 2t + 4u$ is minimal if and only if $u = -1 - \frac{t}{2}$, but we then have $N(e - e_4 - e_6 - e_7) = 1 + \frac{t+v}{2} \le 3$ since $|t|$ and $|v|$ are bounded from above by 2. Hence $s_f = 9$ implies $s' \le 8$. This completes the proof of the proposition. $\square$

Proof of Theorem 1.1 *for dimension* 7. Lemma 4.1 shows that it suffices to consider lattices which are either of index 2 and length 5, or of index 4, cyclic or not. The bound $s \le 17$ for lattices which are not generated by their minimal vectors then results from the three propositions 3.1, 4.3 and 4.4. The fact that there exists such lattices with $s = 17$ is seen on the Gram matrix displayed in the appendix, which defines a lattice having a 5-dimensional section similar to Coxeter's perfect lattice $\mathbb{A}_5^3$ (the perfect lattice P_5^2 in Conway and Sloane's notation; see [3], Table 6.5.5). $\square$

5. Appendix

In this appendix, we display Gram matrices for lattices Λ of dimension $n \le 7$ with given maximal index $\imath$ and length ℓ, having the largest possible kissing number among those which are not generated by their minimal vectors. We discard the case where $n = 7$, $\imath = 2$ and $\ell = 5$, for which the optimal value for s (14, 15, 16 or 17) is not known, see the remark below.

We disregard the trivial case where $\imath = 2$ and $\ell = n$, the minimal class of which (in the sense of [3], Section 9.1) is that of the centred cubic lattice. This is the only possibility if $n = 5$. If $n = 6$, the only other possibility is $\imath = 2$ and $\ell = n - 1 = 5$. If $n = 7$ and $\imath \ge 3$, then $\ell = 7$, and we are left with 5 cases, namely: $\imath = 2$, $\ell = 6$ or 5; $\imath = 3$; $\imath = 4$, cyclic or non-cyclic.

Matrices are constructed starting with the Gram matrix A' of a basis of minimal vectors for a lattice Λ' such that Λ is of the form $\langle \Lambda', e\rangle$ (resp. $\langle \Lambda', e, f\rangle$) and then replacing the first basis vector by e (resp. the first and the last ones by e and f respectively).

We denote by T a set of representatives up to sign for $S(\Lambda) \smallsetminus \{\pm e_i\}$.

Dimension 6. $\imath = 2$, $\ell = 5$; $s_{\max} = 9$. We display two matrices $A6a$ and $A6b$ for the two minimal classes defined by $T = \{e_1 + e_6, e_2 + e_6, e_3 + e_6\}$ and $T =$

$\{e_1+e_6, e_2+e_6, e_1+e_2+e_6\}$ respectively:

$$A6a = \begin{pmatrix} 75 & 36 & 36 & 21 & 21 & -35 \\ 36 & 48 & 12 & 0 & 0 & -24 \\ 36 & 12 & 48 & 0 & 0 & -24 \\ 21 & 0 & 0 & 48 & -6 & 1 \\ 21 & 0 & 0 & -6 & 48 & 1 \\ -35 & -24 & -24 & 1 & 1 & 48 \end{pmatrix}, \quad A6b = \begin{pmatrix} 5 & 2 & 2 & 2 & 2 & -2 \\ 2 & 4 & 0 & 0 & 0 & -2 \\ 2 & 0 & 4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 4 & 0 & 0 \\ 2 & 0 & 0 & 0 & 4 & 0 \\ -2 & -2 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Dimension 7. $\imath = 2$, $\ell = 6$, $T = \{e_1+e_7, \dots, e_6+e_7\}$, $s_{\max} = 13$;
$\imath = 3$, $T = \{e_1-e_2, e_3-e_4, e_5-e_6\}$, $s_{\max} = 10$.

$$A7a2 = \begin{pmatrix} 27 & 9 & 9 & 9 & 9 & 9 & -12 \\ 9 & 8 & 2 & 2 & 2 & 2 & -4 \\ 9 & 2 & 8 & 2 & 2 & 2 & -4 \\ 9 & 2 & 2 & 8 & 2 & 2 & -4 \\ 9 & 2 & 2 & 2 & 8 & 2 & -4 \\ 9 & 2 & 2 & 2 & 2 & 8 & -4 \\ -12 & -4 & -4 & -4 & -4 & -4 & 8 \end{pmatrix}, \quad A7a3 = \begin{pmatrix} 20 & 9 & 9 & 9 & 9 & 9 & 6 \\ 9 & 18 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 18 & 9 & 0 & 0 & 0 \\ 9 & 0 & 9 & 18 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 18 & 9 & 0 \\ 9 & 0 & 0 & 0 & 9 & 18 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 18 \end{pmatrix}.$$

$\imath = 4$. We display two matrices $A7a4$ and $A7b4$ for the two minimal classes defined by $T = \{\frac{e_1 \pm e_2 \pm e_3 \pm e_4}{2}\}$ and non-cyclic quotient, with $s_{\max} = 15$, and $S = S(P_5^2) \cup \{\pm e_6, \pm e_7\}$ and cyclic quotient, with $s_{\max} = 17$, respectively.

$$A7a4 = \begin{pmatrix} 4 & 2 & 2 & 2 & 0 & 0 & 2 \\ 2 & 4 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 4 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 & 2 \\ 2 & 0 & 2 & 2 & 2 & 2 & 5 \end{pmatrix}, \quad A7b4 = \begin{pmatrix} 9 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 8 & 2 & 2 & 2 & 0 & 0 \\ 4 & 2 & 8 & 2 & 2 & 0 & 0 \\ 4 & 2 & 2 & 8 & 2 & 0 & 0 \\ 4 & 2 & 2 & 2 & 8 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 8 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 8 \end{pmatrix}.$$

[In the cyclic case, there might exist other minimal classes with $s = 17$.]

Remark 5.1. Extending A6a to dimension 7 by $e_i \cdot e_7 = -e_i \cdot e_6$ $(i = 1,2,3,4,5)$, $e_6 \cdot e_7 = -24$ and $e_7 \cdot e_7 = 48$, we obtain a lattice of index 2 and length 5 with $s = 14$ and extra minimal vectors $e_i + e_6$, $e_i - e_7$ $(i = 1,2,3)$ and $e_6 + e_7$.

Remark 5.2. Let Λ be an 8-dimensional, well-rounded lattice with $s \geq \frac{n(n+1)}{2} = 36$ and let Λ_0 be the sublattice generated by its minimal vectors. I could show that one of the following conditions holds for Λ:

1. Λ has a basis of minimal vectors.
2. $\imath(\Lambda) = 4$, $\imath(\Lambda_0) = 2$ and $[\Lambda : \Lambda_0] = 2$.
3. $\imath(\Lambda) = 6$, $\imath(\Lambda_0) = 3$ and $[\Lambda : \Lambda_0] = 2$.

I conjecture that assertion (1) is indeed the correct one, and even that the hypothesis $s \geq 36$ could be slightly weakened. This conjecture implies that 8-dimensional perfect lattices have bases of minimal vectors, a result which is actually a consequence of the recent classification theorem of Dutour–Schürmann–Vallentin ([2]).

Lattices of maximal index 2 play a crucial rôle in the remark above. They are of two different types according to whether they are generated or not by their minimal vectors. Recall that for $n \geq 3$ and $r \mid n+1$, the *Coxeter lattice* $\mathbb{A}_n^r$ is the unique lattice L such that $\mathbb{A}_n \subset L \subset \mathbb{A}_n^*$ and $[L : \mathbb{A}_n] = r$.

If $[\Lambda : \Lambda_0] = \imath(\Lambda) = 2$ and $s(\Lambda) \geq \frac{n(n+1)}{2}$, then (by a theorem of Korkine and Zolotareff; see [3], Section 6.1), Λ_0 is similar to the root lattice $\mathbb{A}_n$. A look at the structure of $\mathbb{A}_n$ modulo 2 then shows that Λ is similar to a lattice of the form $\langle \mathbb{A}_m^2, \mathbb{A}_n \rangle$ for some odd $m \leq n$. Now (see [3], Sections 5.1 and 5.2), we have $S(\mathbb{A}_m^2) = S(\mathbb{A}_m)$ if and only if $m \geq 9$. This gives us a complete description of lattices with $\imath(\Lambda) = 2$ which are not generated by their minimal vectors; in particular, they exist if and only if $n \geq 9$.

For *perfect* lattices with $\imath(\Lambda) = 2$ which are generated by their minimal vectors, I only have the following conjecture:

Conjecture 5.3. *A perfect lattice of maximal index* 2 *which is generated by its set of minimal vectors has dimension at most* 7. *(More optimistic form: replace "perfect" by "$s \geq \frac{n(n+1)}{2}$".)*
[In Conway and Sloane's notation (see [3], Section 6.5), the perfect lattices with $\imath = 2$ which are generated by their minimal vectors are P_4^1, P_5^1, P_5^2, P_6^5, P_6^6 and P_7^{32}.]

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