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Complements to
Perfect Lattices in Euclidean Spaces.

Note. Since the publication of *Perfect Lattices in Euclidean Spaces*, improvements on several questions considered in this book appeared in the literature, or have been noticed by various researchers who read my book. Besides the erratum, my home page displays an update of the reference list (completion of the original which appeared after the book as well as new references quoted after the original reference list).

In what follows, we intend to list some results which improve on those which occur in the book, and either give (an outline of) a proof or refer to the updated reference list. The last page, formerly an appendix to the *erratum*, is devoted to Craig lattices.

Section 1.10.C.

Problems involving tensor products of lattices play an interesting rôle in Arakelov theory. The interested reader could look at the following two papers by D. Grayson (Comment. Math. Helvetici 59 (1984),600–634 & 61 (1986), 661–676) and at recent papers by Y. André (Tohoku Math. J. 63 (2011), 629-–649), J.-B. Bost & H.Chen (arXiv:1203.0216v1 [math.NT]), B. Casselman (Asian Journal of Mathematics 8 (2004); issue dedicated to Armand Borel), and É. Gaudron & G. Rémond (arXiv:1109.2812v1 [math.NT]); quoted by Fabien Pazuki in

http://www.math.u-bordeaux1.fr/~fpazuki/GDTGD.html .

Section 2.2.C.

An analogue of the Hermite constant for global function fields is defined in [Hu-Y].

Section 2.3.C.

Generalization of the Mordell inequality to lattices endowed with various algebraic structures have been obtained by Stephanie Vance in [Van1]. In particular, her inequality shows that the maximal Hermite invariant among 16-dimensional lattices over the Hurwitz order is that of the Barnes-Wall lattice Λ_{16}; this makes use of the fact (proved by Sigrist and by Schürmann) that the densest 12-dimensional Hurwitz lattices are $\Lambda_{12}^{\max}$ and $\Lambda_{12}^{\min}$.

Section 2.6.C.

Achill Schürmann pointed out to me that van der Waerden's constants δ_p quoted in Theorem 2.6.11 are indeed equal to $\left(\frac{5}{4}\right)^{p-4}$ for all $p \geq 4$. Alternative reference:

[vdW'], *same title*, Acta Math. **96** (1956), 265–309.

He moreover puts ahead a conjecture according to which the exponential bound above could be replaced for $n \leq 8$ by the linear bound $\frac{n}{4}$. This I have proved in an unpublished preprint (*Hermite versus Minkowski, November, 2007*), now on arXiv as [Mar14].

[Added August 12th, 2021.] I recently noticed the paper [Reg] by S. Regavim (and its joint appendix with L. Hadassi) which throws new light on reduction problems for lattices. What concerns Schürmann's conjecture (around van der Waerden's theorem) has been previously solved up to dimension 8 in [Mar14] in a somewhat crude form. However the results of Regavim for dimensions 6 and 7 are more precise.

The methods of [Mar14] can probably be pushed to dimension 9 using [Mar-Schr1], and maybe to dimensions 10 and 11, using unpublished data of M. Dutour-Sikirić, but do not look suitable to attack the problem in large dimensions.

Regavim's paper reminds me of the existence of the paper [L-L-S], which appeared between the French 1996 and English 2003 editions of my book, and that I should have quoted in my printed book.

Section 2.8.C.

C. Poor and D. Yuen have obtained the exact values of γ'_n and the corresponding dual-critical lattice for $n = 5, 6, 7$; see Section 6.4.C below.

[November 14th, 2008.] In a mail dated October 23rd, 2008, T. Watanabe wrote: "In the last Monday , we have found that Hermite-Rankin constants of 8-dimensional are immediately determined by the Bergé-Martinet constants of 5 and 7 dimensional, which were determined by Poor and Yuen."; see [S-Wt-O]; the "Bergé-Martinet" inequalities are those of [B-M1] and are accounted for in this section 2.8.

(Feb. 19, 2009). Recent progress has been made on the constant γ''_n introduced in [B-M1].

(1) On request of the editorial board of "J. Th. Nombres de Bordeaux", a paper by Marc Gindraux ([Gi'], replacing the inaccessible [Gi] quoted in the book) has been published in the third issue of 2009 of the journal. This is a compact form of a 2002 text intended to be part of a Ph. D., which was never completed (the author no longer

works in a university). The constant γ''_n is broken into two constants $\gamma''_{n,+}$ and $\gamma''_{n,-}$ in order to take care of problems of non-continuity; see "publications by various authors" in my homepage.

(2) The paper [Pe-vZ] by R.A. Pendavingh et S.H.M. van Zam gives a very sharp upper bound for γ''_5, which strongly suggests that the exact value is attained on Blichfeldt's example.

Section 3.2.C.

Definition 3.22 (3) could be enlarged by introducing the notion of strong semi-eutaxy:

Definition 3.2.C1. *We say that a system of lines or a lattice is* strongly semi-eutactic *if it is semi-eutactic with equal nonzero coefficients.*

It is easily checked that the set of lines (or vectors) having nonzero corresponding eutaxy coefficients is then a 3-design in the sense of Chapter 16. The first example occurs in dimension 4; see the file *strongeut.gp* on my home page, which contains the complete classification of strongly (semi-)eutactic lattices up to dimension 6. A more "classical" example is provided by ${K'_9}^*$; see Section 8..

In the course of the joint research [B-M8] with Anne-Marie Bergé, we noticed the following result on weakly eutactic lattices, which improves Corollary 3.2.6 for such lattices:

Proposition 3.2.C2 (June, 2005). *Let Λ be weakly eutactic and let* $\mathrm{Id} = \sum_{s\in S(\Lambda)/\pm} \rho_x\, p_x$ *be a eutaxy relation for Λ. Then the set of minimal vectors of Λ having strictly positive coefficients ρ_x spans E.*

Proof. Let $T = \{x \in S \mid \rho_x > 0\}$ and let $F \subset E$ be the span of T. Let $y \in E$ be orthogonal to all vectors of T. Let us apply to y the eutaxy relation above. Since $p_x(y) = \dfrac{1}{\min\Lambda}\,(x\cdot y)\,x$, this relation reads $y = \dfrac{1}{\min\Lambda}\,\sum_{\rho_x<0} \rho_x\,(x\cdot y)\,x$. Taking the scalar product of both sides with y, we now obtain $N(y) = \dfrac{1}{\min\Lambda}\,\sum_{\rho_x<0} \rho_x\,(x\cdot y)^2$.

Since the left hand side is ≥ 0 and the right hand side is ≤ 0, both are zero. In particular, we have $N(y) = 0$, i.e. $y = 0$. $\square$

SECTION 3.5.C.

The characterization given in Proposition 3.5.3, (3) of perfection for lattices having a perfect hyperplane section with the same minimum has the following easy generalization: $\operatorname{perf}\Lambda = \frac{n(n-1)}{2} + \operatorname{rk} S \setminus S \cap H$ (where perf stands for the perfection rank). This formula is itself a special case of the following more general statement:

Proposition 3.5.C3. *Let S be a (finite) subset of E, let H be a hyperplane in E, and let $S_0 = S \cap H$. Denote by r_0 the perfection rank of S_0, by r that of S, and by t the* rank *of $S \setminus S_0$. Then we have the inequality $r \geq r_0 + t$, and equality holds whenever either S contains exactly t pairs of vectors off S_0, or S_0 is perfect.*
[This applies to a lattice Λ and a hyperplane section Λ_0 of Λ, *assuming that* $\min \Lambda_0 = \min \Lambda$.]

Proof. We normalize all vectors of S to norm 1, and denote by e a unit vector orthogonal to H.

First observe that vectors $x_1, \dots, x_k \in S \setminus S_0$ are dependent if and only if there exists a relation $\sum \lambda_i\, p_{x_i} = 0$ in $\operatorname{End}^s(E)/\operatorname{End}^s(H)$ with coefficients not all zero off H. Indeed, a relation $\sum_x \lambda_x\, p_x = 0$ in $\operatorname{End}^s(E)$ implies $\sum_{x \notin H} \lambda_x\, p_x \circ p_e$ because $p_x \circ p_e = 0$ on H. Applied to e, it reads $\sum_{x \notin H} \lambda_x\, (x \cdot e)\, x = 0$, a non-trivial dependence relation since $x \cdot e \neq 0$ off H. Conversely, a dependent relation can be written $\sum_{x \notin H} \lambda_x\, (x \cdot e)\, x = 0$, and implies $\sum_x \lambda_x\, p_x \in \operatorname{End}^s(H)$, since for all $y \in E$, we have

$$\textstyle\sum_x \lambda_x\, (p_x \circ p_e)(y) = (e \cdot y) \sum_x \lambda_x\, (x \cdot e)\, x\,.$$

Now choose t independent vectors $y_1, \dots, y_t \in S \setminus S_0$ and vectors $x_1, \dots, x_{r_0} \in S_0$ such that the p_{x_i} constitute a basis for the span of the $p_x, x \in S_0$. Then the projections p_{x_i}, p_{y_j} are independent, which shows that $r \geq r_0 + t$.

If $S \setminus S_0$ reduces to the $\pm x_i$, then the vector space $\{p_x \mid x \in S\}$ is generated by the p_{x_i}, p_{y_j}, whence the inequality $r \leq r_0 + t$ in this case.

If S_0 is perfect, then the p_x, $x \in S_0$ span $\operatorname{End}^s(H)$, so that the p_x, $x \in S \setminus S_0$ are again linear combinations of the p_{x_i} and the p_{y_j}. $\square$

The notion of a *perfection relation* occurs only incidentally in the book (in Remark 6.2.5.). A formal definition should have been given in this section: a *perfection relation on a lattice* Λ is a relation of the form

$$\textstyle\sum_{x \in S(\Lambda)/\{\pm\}} a_x\, p_x$$

with real coefficients a_x. In terms of matrices, this takes the form $\sum_{X\in S(M)/\{\pm\}} a'_X\, X^tX$, with set of coefficients (a'_X) proportional to (a_x). Taking the trace shows that the a_x add to zero.

The set of perfection relations is a real vector space of dimension $s-r$ (s is the kissing number, r is the perfection rank); and the set of eutaxy relations, if non-empty, is an affine space over the space of perfection relations. More details can be found in the paper [B-M10] of the complementary reference list.

Section 3.8.C.

The first part of Corollary 3.8.6 could have been more suitably stated in a form which separates the properties of (dual-) perfection and eutaxy. For a perfect lattice, we can even prove slightly more.

Corollary 3.8.C1. *Let Λ be a eutactic (resp. an extreme) lattice. If Λ^* is eutactic (resp. semi-eutactic), then Λ is dual-eutactic (resp. dual-extreme).*

Proof. Since Λ^* is semi-eutactic, there exists for Λ^* a eutaxy relation of the form $\mathrm{Id} = \sum_{y\in S_1/\pm} \rho_y\, p_y$ with strictly positive coefficients ρ'_y for some symmetric subset S_1 of $S(\Lambda^*)$. Eliminating the identity in this relation and in a eutaxy relation for Λ, we obtain a relation of dual-eutaxy of the form

$$\sum_{x\in S(\Lambda)/\pm} \rho_x\, p_x = \sum_{y\in S_1/\pm} \rho'_y\, p_y$$

with strictly positive coefficients ρ_x, ρ'_y.

If $S_0 = S(\Lambda^*)\setminus S_1$ is empty, this shows that Λ is dual-eutactic.

Otherwise, we may assume that Λ is extreme. Then the endomorphism $\sum_{z\in S_0/\pm} p_z$ is a combination $\sum_{x\in S(\Lambda)} \mu_x\, p_x$. For $\varepsilon > 0$ small enough, the relation

$$\sum_{x\in S(\Lambda)/\pm} (\rho_x + \varepsilon\mu_x)\, p_x = \sum_{y\in S_1/\pm} \rho_y\, p_y + \sum_{z\in S_0/\pm} \varepsilon\, p_z$$

is a relation of dual-eutaxy with strictly positive coefficients. $\square$

[The argument indeed shows that if Λ is perfect and possesses a relation of dual-eutaxy $\sum_{x\in S(\Lambda)} \rho_x\, p_x = \sum_{y\in S(\Lambda^*)} \rho'_y\, p_y$ with strictly positive ρ_x and non-negative ρ'_y, then Λ is dual-extreme.]

In general, dual-eutaxy does not imply any eutaxy property. However, the following statement establishes a partial converse to the corollary above:

Proposition 3.8.C2. *Let Λ be a eutactic and dual-eutactic lattice. Assume that its automorphism group acts transitively on its set of minimal vectors. Then Λ^* is eutactic.*

Proof. Consider a relation of dual-eutaxy

$$\sum_{x \in S(\Lambda)/\pm} \rho_x \, p_x = \sum_{y \in S(\Lambda^*)/\pm} \rho'_y \, p_y$$

with strictly positive coefficients ρ_x, ρ'_y. By averaging on the automorphism group, we obtain a new relation of the form

$$\sum_{x \in S(\Lambda)/\pm} p_x = \sum_{y \in S(\Lambda^*)/\pm} \rho''_y \, p_y$$

with strictly positive coefficients ρ''_y. By Theorem 3.6.6, Λ is strongly eutactic. This shows that the left hand side of the equality above is proportional to the identity, and so is the right hand side. $\square$

The mere condition that Λ should be strongly eutactic does not imply that Λ^* is eutactic. An example is provided by the 9-dimensional lattice Λ found by Baril and referred to in Section 14.5, for which $\gamma'^2 = \frac{16}{5}$, the largest known value in dimension 9 (also attained on Coxeter's $\mathbb{A}_9^2$). Here is a Gram matrix for Λ; see also *strongeut.gp*, matrix *baril9*:

$$\begin{pmatrix} 6 & -2 & 2 & -1 & -2 & 0 & -2 & -1 & 2 \\ -2 & 6 & -2 & 2 & -1 & -2 & 0 & -2 & -1 \\ 2 & -2 & 6 & -2 & 2 & -1 & -2 & 0 & -2 \\ -1 & 2 & -2 & 6 & -2 & 2 & -1 & -2 & 0 \\ -2 & -1 & 2 & -2 & 6 & -2 & 2 & -1 & -2 \\ 0 & -2 & -1 & 2 & -2 & 6 & -2 & 2 & -1 \\ -2 & 0 & -2 & -1 & 2 & -2 & 6 & -2 & 2 \\ -1 & -2 & 0 & -2 & -1 & 2 & -2 & 6 & -2 \\ 2 & -1 & -2 & 0 & -2 & -1 & 2 & -2 & 6 \end{pmatrix}$$

This lattice is extreme and strongly eutactic, with $s = 45$; its dual, which has $s = 40$, is not even weakly eutactic. The automorphism group of Λ acts transitively on $S(\Lambda^*)$, but has two orbits S_1 with $s_1 = 5$ and S_2 with $s_2 = 40$ on $S = S(\Lambda)$. Dual eutaxy can be studied using the averaging argument of Proposition 3.8.8: if Λ is dual-eutactic, there exists a relation of dual eutaxy involving sums of projections to vectors of S_1, S_2 and S^*. Using formula 3.8.3', we check that such a relation must be unique up to proportionality and that

$$8 \sum_{x \in S_1/\pm} p_x + 3 \sum_{x \in S_2/\pm} p_x = 4 \sum_{y \in S^*/\pm} p_y$$

indeed holds.

An attempt for classifying dual-extreme lattices in dimension 5 was done by Anne-Marie Bergé at the time the English edition of this book was prepared. Her aim was to prove that γ'_5 is attained uniquely on

the two perfect lattices P_5^1 and P_5^2 and their duals, at a time when Poor-Yuen's result was no known. I briefly explain her ideas below.
Her basic idea consists in looking at putative dual-extreme lattices in each minimal class. A lattice with perfection rank $r = 15$ is extreme and dual-extreme (there are three such lattices, namely $\mathbb{D}_5$, $\mathbb{A}_5^3$, and $\mathbb{D}_5$). We now consider classes and lattices with $s \leq 14$. Next observe that if a 5-dimensional lattice has one (resp. at least two) perfect $\mathbb{D}_4$ minimal sections, then it has $s^* = 1$ (resp $s \geq 12 + 12 - 6 = 18$, hence is similar to $\mathbb{D}_5$), so that we may restrict ourselves to lattices without a $\mathbb{D}_4$ minimal sections, that is, lattices with $r = s$. Since we have $s + s^* \geq \frac{n(n+1)}{2} + 1 = 16$, we may restrict ourselves to classes with $r \geq 8$. Using essentially computations by hand, she succeeded in dealing with all classes of perfection rank $r \geq 11$. She PROBABLY proved more, namely that *there does not exist dual-extreme lattices with perfection rank* $r = 14, 13, 12$ *or* 11.

Up to similarity, lattices in a class with perfection rank r depend on $15 - r$ parameters, so that the difficulty of the task is a rapidly increasing function of the co-rank $15 - r$. To go further, and to check A.-M. Bergé's results, it would be highly desirable to make algorithmic her procedure.

SECTIONS 4.2.C TO 4.5.C (September 202)

The knowledge of the sets of orbits of *pairs* of minimal vectors having a given scalar products is useful is some applications (e.g., to the theory of graphs). This is done in the book for the lattice E_8. Here is the general result:

Proposition. *Let Λ be an irreducible root lattice. Then the ordered pairs (x, y) of minimal vectors of Λ having a given scalar product constitute a unique orbit under* $\mathrm{Aut}(\Lambda)$ *except for $\mathbb{D}_n$, $n \geq 5$ where there are two orbits if $x \cdot y = 0$, with representatives $(\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2)$ and $(\varepsilon_1 + \varepsilon_2, (\varepsilon_3 + \varepsilon_4)$.*

Sketch of proof. By transitivity we may choose x arbitrarily. We choose $x = \varepsilon_0 - \varepsilon_1$ for $\mathbb{A}_n$ and $x = \varepsilon_1 + \varepsilon_2$ for $\mathbb{D}_n$ and $\mathbb{E}_n$. It suffices to consider the cases $x \cdot y = 1$ and $x \cdot y = 0$.

$\mathbb{A}_n$. Then $x \cdot y = 1$ (resp. $x \cdot y = 0$) amounts to $y = \varepsilon_0 - \varepsilon_i$ or $-\varepsilon_i + \varepsilon_i$, $i \geq 2$ (resp. $y = \pm(\varepsilon_i - \varepsilon_j)$, $2 \leq i < j \leq n$), and under the fixator G_1 of x in $\{\pm \mathrm{Id}\} \times S_{n+1}$, y is clearly equivalent to $\varepsilon_0 - \varepsilon_2$ (resp. to $\varepsilon_2 - \varepsilon_3$).

$\mathbb{D}_n$. We consider the subgroup $G = \{\pm 1\} \times S_n$ of $\mathrm{Aut}(A_n)$ and the fixator G_1 of x in G. Then $x \cdot y = 1$ amounts to $y = \varepsilon_1 \pm \varepsilon_i$ or $\varepsilon_2 \pm \varepsilon_i$, $i \geq 3$, and y is clearly equivalent to $\varepsilon_1 + \varepsilon_3$ under G_1.

However, $\pm(\varepsilon_1 - \varepsilon_2)$ and $\pm\varepsilon_i \pm \varepsilon_j$, $3 \le i < j \le n$ belong to different orbits under G_1, those of $\varepsilon_1 - \varepsilon_2$ and $\varepsilon_1 + \varepsilon_3$. Since G is the whole automorphism group of $\mathbb{D}_n$ for $n \ge 5$, we are done if $n \ge 5$. If $n = 4$ we identify $\mathbb{D}_4$ with the ring $\mathfrak{M}$ of Hurwitz quaternions equipped with the form $\operatorname{Trd}(x\overline{y})$ and the basis (ε_ℓ) to $(1, i, j, k)$. Then right multiplication with $\omega := \frac{-1+i+j+k}{2}$ transforms $(1+i, 1-i)$ into $(-1+k, i+j)$ belonging to distinct orbits.

$\mathbb{E}_n$, $n = 8, 7, 6$. Let $e = \frac{(\varepsilon_1-\varepsilon_2)+vp_3+\varepsilon_4+\varepsilon_5+(\varepsilon_6-\varepsilon_7+\varepsilon_8)}{2}$ and $e' = e + \varepsilon_2$. These are elements of $\mathbb{E}_6$ and we have $\mathbb{E}_8 = \mathbb{D}_8 \cup (e + \mathbb{D}_8)$. Under the action of the Weyl group $W(\mathbb{D}_8) \cap \mathbb{E}_n$, vector $y \in \mathbb{E}_n$ with $x \cdot y = 1$ (resp. $x \cdot y = 0$) are equivalent to $\varepsilon_1 + \varepsilon_3$ or e' (resp. $\varepsilon_1 - \varepsilon_2$, $\varepsilon_3 + \varepsilon_4$ or e). Using the formulae displayed in the proof of Theorem 4.4.4, we check that the symmetry along e send $\varepsilon_1 + \varepsilon_3$, $\varepsilon_1 - \varepsilon_2$ and $\varepsilon_3 + \varepsilon_4$ to elements of $e + \mathbb{D}_8$, which completes the proof of the proposition. $\square$

Section 4.C. Exercises. (July, 2010; January, 2012.)

I should have added a useful third question to Exercise 4.3.1, and written some more exercises on vectors having a given norm in root lattices.

Exercise 4.3.1... 3. Denote by $\mathbb{D}_2$ the embedding of $\mathbb{A}_1 \perp \mathbb{A}_1$ of the form $\langle \varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j \rangle$ and by $\mathbb{D}_3$ that of $\mathbb{A}_3$ of the form $\langle \varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_k, \varepsilon_j + \varepsilon_k \rangle$. Show that every root sublattice of rank n of $\mathbb{D}_n$ is isometric to an orthogonal sum of lattices $\mathbb{D}_{m_i}$ with $m_i \ge 2$ and $\sum_i m_i = n$, and is well defined up to an automorphism of $\mathbb{D}_n$ by the set $\{m_i\}$.

Nota bene. For the complementary exercises for Chapter 4 below, the reader could look at my papers [Mar8] and [Mar10] of the extended reference list on the homepage.

Note in particular that for any n-dimensional lattice of minimum m, if two vectors x, y of norms $N \le 2m$ define the same class modulo 2, then either $y = \pm y$, or $N(y) = N(x) = 2m$ and $y \cdot x = 0$; in the latter case, for a given x, the set of vectors $\frac{y \pm x}{2}$ with $N(y) = 4$ and $y \equiv x \mod 2$ constitute a root system of type $\boldsymbol{D}_k$ rescaled to norm m ($k = k(x)$ may depend on x). Other results for classes modulo 2 or 3 can be found in [Mar8] or [Mar10].

Exercise 4.C1. 1. Show that the vectors of norm 4 in an exceptional lattice Λ constitute a single orbit under $\operatorname{Aut}(\Lambda)$, containing 135 pairs of vectors if $\Lambda \simeq \mathbb{E}_6$, 378 if $\Lambda \simeq \mathbb{E}_7$, and 1080 if $\Lambda \simeq \mathbb{E}_8$.

2. Show that the set of norm 4 vectors congruent to one of them modulo 2 consists of k pairwise orthogonal pairs of vectors with $k = 5$ if $\Lambda \simeq \mathbb{E}_6$, $k = 6$ if $\Lambda \simeq \mathbb{E}_7$, and $k = 8$ if $\Lambda \simeq \mathbb{E}_8$. [One can choose as representatives under $\operatorname{Aut}(\Lambda)$ the system $\pm 2\varepsilon_i$ with $1 \le i \le 5$, $1 \le i \le 6$, and $1 \le i \le 8$, respectively.]

3. Show that $s_2 + \frac{s_4}{k}$ is equal to $2^6 - 1$ if $\Lambda \simeq \mathbb{E}_6$, to $2^8 - 1$ if $\Lambda \simeq \mathbb{E}_8$, and to $2^8 - 2$ if $\Lambda \simeq \mathbb{E}_7$.

4. Show that if if $\Lambda \simeq \mathbb{E}_6$ or $\Lambda \simeq \mathbb{E}_8$, all classes of Λ modulo 2 have representatives of norm $N \leq 4$, but that there is one missing class if $\Lambda \simeq \mathbb{E}_7$.

Exercise 4.C2. (The missing class in $\mathbb{E}_7$ modulo 2.) 1. Show that the norm 6 vector $y = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 \in \mathbb{E}_7$ is not congruent to a vector of smaller norm.

2. Show that the class $\mathcal{C}_x$ of x *modulo* 2 consists of
(a) the 16 pairs $\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6$ having an even number of minus signs, and
(b) the 12 pairs $\varepsilon_7 - \varepsilon_8 + 2\varepsilon_i$, $i = 1, 2, 3, 4, 5, 6$.

3. Show that $C_x = 2\,S(\mathbb{E}_7^*)$.

4. Prove that there are 1008 pairs of norm-6 vectors off $2S(\mathbb{E}_7^*)$, that they constitute one orbit under $\mathrm{Aut}(\mathbb{E}_7)$, and that every minimal vector of $\mathbb{E}_7$ is congruent to 16 such pairs of vectors. (See Exercise 4.4.9.)

Exercise 4.C3. 1. Show that the automorphism group of $\mathbb{E}_6$ acts transitively on its norm 6 vectors.

2. Show that the norm 6 vectors of the class $\mathcal{C}$ of $\varepsilon_1 + \varepsilon_2$ consists of
(a) the 6 pairs $\varepsilon_1 - \varepsilon_2 \pm 2\varepsilon_i$, $i = 3, 4, 5$, and
(b) the 4 pairs $\pm\varepsilon_3 + \pm\varepsilon_4 + \pm\varepsilon_5 + \pm(\varepsilon_6 - \varepsilon_7 + \varepsilon_8)$ having an even number of minus signs.

3. Show that the configuration of these 10 pairs of vectors is that of $S(\mathbb{A}_5^2)$. (The Coxeter lattice $\mathbb{A}_n^2$ is defined in Section 5.2.)

Exercise 4.C.4. E_7^* modulo 2 (*scaled to minimum* 3).

1. Show that the norm of $x \in \mathbb{E}_7^*$ is of the form $4k + 3$ if $x \in \mathbb{E}_7^* \setminus \mathbb{E}_7$ and $4k$ if $x \in \mathbb{E}_7$, $k \geq 0$.

2. Show that there is one orbit of vectors for each of the norms 3, 4, 7, having 28, 63 and 288 pairs of vectors, respectively.

3. Prove that vectors of norm 7 appear in sets of 8 pairs $\pm x$ consisting of one class modulo $2\,\mathbb{E}_7^*$, with configuration $\mathbb{A}_7^*$.

4. Prove that these vectors represent all non-zero classes modulo 2, according to the formula $28 + 63 + \frac{288}{8} = 2^7 - 1$.

Section 5.1.C.

Not to assume that r is integral is useless!

SECTION 5.2.C.

A way of using Coxeter lattices to increase $\gamma(\mathbb{A}_n)$ is to consider lattices $\langle \mathbb{A}_m^r, \mathbb{A}_n \rangle$ for $m \le n$ and $r \mid m+1$. In particular lattices with $r \mid n$ of Proposition 5.2.5 are isometric to $\langle \mathbb{A}_{n-1}^r, \mathbb{A}_n \rangle$. This applies notably to the dense lattices $\mathbb{A}_8^2$ and $\mathbb{A}_9^3$ and shows immediately the existence of cross-sections $\mathbb{E}_7$ and $\mathbb{E}_8$, respectively.

One can even insert in $\mathbb{A}_n$ two or more copies of Coxeter lattices of lower dimensions provided n is large enough.

SECTION 5.3.C. (May 2nd, 2015)

The way we construct the lattices $P_n = \mathbb{A}_n^{(2)}$ in this section is a special case of the following general procedure. Consider an n-dimensional lattice Λ with successive layers of norm $m_0 = 0$, $m_1 = \min \Lambda$, $m_2, m_3, ...$, a finite additive Abelian group A of order m, and a surjective homomorphism $\varphi : \Lambda \to A$. Then $L := \ker \varphi$ is a lattice of index m^2 in Λ, hence $\det(L) = m^2 \det(\Lambda)$. If φ is non-zero on $S(\Lambda)$ we have $\min L \ge m_2$. In case of equality (which is in practice the usual situation) we have $\gamma(L) = \gamma(\Lambda) \times \dfrac{m_2}{m_1} m^{-2/n}$. Otherwise we shall have a multiplicator $\frac{m_3}{m_1} m^{-2/n}$, etc. When A is cyclic, L can be viewed as the lattice orthogonal *modulo* m to a convenient vector of Λ^*.

Taking $\Lambda = \mathbb{A}_n$ equipped with its Korkine-Zolotareff basis (e_i), we must get rid of the vectors e_i and $e_i - e_j, i < j$, which needs $\varphi(e_i) \ne 0$ and $\varphi(e_j) \ne \varphi(e_i)$ for $i < j$, hence $|A \setminus \{0\}| \ge n$, i.e., $m \ge n+1$; and if $m = n+1$, all choices of φ are equivalent under the automorphisms of $\mathbb{A}_n$ induced by permutations of the e_i; for instance, we may take $\varphi(e_i) = i \bmod m$, a map induced by $\varphi(\varepsilon_i) = i$ on the canonical basis $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ of $\mathbb{Z}^{n+1}$.

Taking $A = \mathbb{Z}/m\mathbb{Z}$ we obtain Barnes's lattices P_n. A generalization of this construction yields the Craig lattices $\mathbb{A}_n^{(r)}$ of minimum $\ge m_r$, a lower bound which is sometimes strict, as one can see in the appendix.

I could have made more precise Proposition 5.3.6.

Proposition 5.3.6.C. For $n \ge 11$, the group G_n, of order $2(n+1)\varphi(n+1)$, is the full automorphism group of P_n.

A similar construction can be done with $\mathbb{D}_n$. The smallest possible modulus is $m = 2n - 2$, which can be achieved taking for φ the restriction of the function defined on $\mathbb{Z}^n$ by $\varphi(\varepsilon_i) = i-1 \mod m$. **Conjecturally**, these lattices are perfect for $n \ge 9$, satisfy $S(L^*) = S(\mathbb{D}_n^*)$ $[= S(\mathbb{Z}^n)]$ for $n \ge 14$, and are never strongly eutactic. I have not tested them for eutaxy.

Constructing the even part of an odd lattice is also a construction of type "$\ker\varphi$"; other such constructions occur in further chapters (e.g., modulo 3 in Section 8.4).

Section 5.4.C.

Variants of the Craig lattices and analogues defined using ideals $\mathfrak{p}_p^i\mathfrak{p}_q^j$ in cyclotomic fields $\mathbb{Q}(\zeta_{pq})$ (p, q primes) have been considered by Flores *et al.* in [F-I-P] and [FIPLO]. A different kind of generalization has been considered by Hao Chen in [Chen].

Section 6.4.C.

In lemma 6.4.13, the ten minimal vectors (up to sign) outside Λ' are e, $e - e_i$ $(i = 1, 2, 3)$, and $e - e_j - e_k$ $(j = 1, 2, 3; k = 4, 5)$. The matrix B_2 corresponds to the more symmetric choice $e - e_i - e_j$, $1 \le i < j \le 5$.

[January 23rd, 2007.] Conjecture 6.4.16 (and the natural ones for dimensions 6 and 7) has been solved by Cris Poor and David Yuen ([P-Y4], 2006): γ'_5 is attained exactly on $\mathbb{D}_5$, $\mathbb{D}_5^*$, $\mathbb{A}_5^3$, $\mathbb{A}_5^{3*} = \mathbb{A}_5^2$, γ'_6 on $\mathbb{E}_6$ and $\mathbb{E}_6^*$, and γ'_7 on $\mathbb{E}_7$ and $\mathbb{E}_7^*$. We have

$$(\gamma'_5)^2 = 2\,, \quad (\gamma'_6)^2 = \tfrac{8}{3} \quad \text{and} \quad (\gamma'_7)^2 = 3\,.$$

Their proofs make use of a new function satisfying a convenient convexity property (rather "concavity"), a *type one function.* Curiously, proofs are shorter in dimensions 6 and 7, in relation with the fact that the two densest perfect lattices in these dimensions are dual to each other. In dimension 5, the proof is by inspection of the minimal classes (classified by Batut), with however an extra trick thanks to which they avoid the consideration of all cases.

Section 6.6.C.

In this overview of our knowledge on 8-dimensional perfect lattices, I mentioned the problem of eutaxy only in the case of Laïhem lattices, which had been dealt with by Jaquet ([Ja7]) a few months after Laïhem's thesis had been completed. Cordian Riener ([Rie]) has recently established the status of the 10916 perfect lattices. His data confirm Jaquet's result. We summarize them in the table below:

Eutaxy for Perfect 8-Dimensional Lattices

Type	Laïhem	Baril	Napias	Batut	total
Eutactic	383	16	1929	66	2394
Semi-eut.	21	0	5	2	28
Weakly eut.	771	37	7608	78	8494
Total	1175	53	9542	146	10916

In the table above, "semi-eutactic" means "semi-eutactic, non-eutactic"; similarly, "weakly eutactic" means "weakly eutactic, non-semi-eutactic".

Among these 2394 eutactic lattices, only 7 are strongly eutactic, namely the Laïhem lattices lh(2), lh(3), lh(8), lh(271), lh(1172) $\simeq \mathbb{E}_8$, lh(1174) $\simeq \mathbb{D}_8$, and lh(1175) $\simeq \mathbb{A}_8$. Their duals are also strongly eutactic except lh(3), for which $s^* = 3$, and lh(8).

We now consider the property of being dual-extreme (or dual-eutactic, this amounts to the same since these lattices are perfect). A necessary condition is that $S(\Lambda^*)$ should have rank 8. This condition is satisfied by only 8 lattices: the 7 Laïhem lattices lh(2), lh(6), lh(8), lh(271), lh(1172), lh(1174), and lh(1175), and the Napias lattice *nap*(6920). Since the duals of 5 of them are strongly eutactic (see above), we are left with lh(6), the dual of which is only (strongly) semi-eutactic, and lh(8) and *nap*(6920), the duals of which are not weakly eutactic. We have calculated the orbit structures of (Λ, Λ^*) for these three lattices: $(3,2)$, $(3,1)$ and $(5,1)$ respectively, and applied the method used in Section 3.8.C above to deal with the 9-dimensional Baril lattice. The conclusion is that lh(6) and lh(8) are dual-extreme and that *nap*(6920) is not. (For lh(6), one can use Corollary 3.8.C1.)

Work by Mathieu Dutour, Achill Schürmann and Frank Vallentin (e-mail from A. Schürmann, June 30th, 2005) has thrown new light on dimension $n = 8$ (and also $n = 9$, for which they found more than 500 000 lattices). In particular, they showed that:

1. All neighbours of the known perfect lattices except perhaps $\mathbb{E}_8$ belong to the list of previously known lattices.

2. Every lattice of the list is a neighbour of some lattice of the list with kissing number $s \leq 39$.

3. Every lattice of the list except $\mathbb{A}_8$ and nap(7014) is a neighbour of $\mathbb{E}_8$.

In an e-mail dated October 5th, 2005, Schürmann wrote "*... we finally managed, with the help of Mathieu Dutour's code for the so called* adjacency decomposition method, *to finish the classification of*

8-*dimensional perfect forms.*" The corresponding manuscript now exists on *arXiv, Article math.NT/0609388, Sept. 18, 2006* ([D-S-V1]). This work proves that the list of 10916 perfect, 8-dimensional lattices referred to in my book (and for which Gram matrices can be downloaded from my home page) *is complete.* In particular, Conjecture 6.6.7 is true (all perfect 8-dimensional lattices have a basis of minimal vectors). The classification also shows that all perfect 8-dimensional lattices have a hexagonal section with the same norm; in particular, all have an even minimum in any scale which make them integral.

[Integral perfect lattices with an odd minimum exist in dimension $n = 1$, $n = 7$ and also $n \geq 10$ by [B-M7], and do not exist for $n = 2, 3, 4, 5, 6$ and 8. The case of dimension 9 remains open. However, Riener checked that the (more than) 500,000 perfect 9-dimensional lattices quoted above have an even minimum in any scale which make them integral. Anne-Marie Bergé and myself *have conjectured that this is general, and even that all perfect* 9-*dimensional lattices do have hexagonal sections with the same minimum.*]

[updated on March 21st, 2023.] Over *two billions* of perfect lattices in dimension 9 have been found by van Woerden ([vW1]). Also lower and upper bounds for the number of perfect lattices in a given dimension have been obtained by Bacher ([Bc5], 2017) and by van Woerden ([vW2], 2018).

[updated on September 16th, 2024.] I just learn by an e-mail of W. van Woerden that he has finished the classification of 9-dimensional perfect form. Alltogether there are

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such forms. They are generated by their minimal vectors *except one of them*, namely a foem for for $\mathbb{A}_9^3$, indeed the only lattice connected uniquely witk Λ_9.

Section 7.4.C.

For relations between the Voronoi graph and minimal classes, see Section 9.1.C below.

Section 8.4.C.

In Theorem 8.4.2, for $n = 2h + 1$ odd, we have $s(L_n) = \frac{9h(h-1)}{2} + \frac{3h}{2}$. This accounts for the existence of two orbits of minimal vectors, that of the section L_{n-1}, and its complementary set in $S(L_n)$.

Section 8.7.C.

(a) See the erratum.

(b) January 12*th*, 2023. The lattices K'_n, $n \leq 24$ are defined in dimensions $n \leq 12$ inside $K'_{12} = K_{12}$, then by symmetry in dimensions $24 - n$.

They differ from laminated lattices only if $3 \leq n \leq 21$. One can define K'_{21} as the orthogonal of a minimal square in Λ^*_{23} (using instead a minimal hexagonal lattice yields Λ_{21}). An alternative construction is as follows: taking the orthogonal of a norm 6 vector in Leech's Λ_{24} we obtain a lattice Λ'_{23} well-defined up to isometry; we then obtain by "antilaminations" first a lattice Λ'_{22}, next a lattice $\Lambda'_{21} \simeq L'_{21}$.

The lattices Λ'_{23} and Λ'_{22} are perfect, strongly eutactic, and have strongly eutactic duals, hence are extreme and dual-extreme,

Section 9.1.C.

After the proof of Theorem 9.1.5, I remark that *perfect classes* (those which contain a perfect lattice, i.e., those of maximal perfection rank $\frac{n(n+1)}{2}$) reduce to the similarity class of *one* perfect lattice. Here I should have also remarked that classes of rank $\frac{n(n+1)}{2} - 1$ are represented by Voronoi paths.

Voronoi paths are listed in Section 6.5. There is *one* path per orbit of their perfect endpoints. *However it may happen that two distinct orbits define equivalent minimal classes.*

(March 20th, 2023.) In dimensions $n \leq 7$, this occurs exactly with the two paths $(i, j) = (1, 9)$ and $(1, 10)$, two out of the eleven paths P_7^1—P_7^1. What happens for $n = 8$is not known.

A more convenient description of the Voronoi graph would be to classify the edges by their minimal class rather than by isometries of their endpoints.

Section 9.4.C.

In the joint research with A.-M. Bergé referred to above in connection with Corollary 3.2.6, we were lead to use a stronger form of Corollary 9.4.2. Here is a more precise statement:

Proposition 9.4.C1 (June, 2005). *On a minimal class $\mathcal{C}$ consisting of lattices whose minimal vectors do not span E (a "non-well–rounded class"), the Hermite invariant has no maximum nor minimum. Moreover,* $\inf_{\Lambda \in \mathcal{C}} \gamma(\Lambda) = 0$.

Proof. Only the "moreover" part needs a proof. Let $\Lambda \in \mathcal{C}$, let $F \subset E$ be the span of $S(\Lambda)$, and let $r = \dim F$. For $\lambda \geq 1$, let u_λ be the linear map which is the identity on F and multiplication by λ on $F^\perp$, and set $\Lambda_\lambda = u_\lambda(\Lambda)$. For $x \in E$ with components y on F and z on $F^\perp$, we have $u_\lambda(x) = x + \lambda y$, hence $N(u_\lambda(x)) = N(x) + \lambda^2 N(y) \geq N(x)$. Hence $S(\Lambda_\lambda) = S(\Lambda)$, so that Λ is still in $\mathcal{C}$. But $\det(u_\lambda) = \lambda^{n-r}$, hence

$$\gamma(\Lambda_\lambda) = \gamma(\Lambda) \cdot \lambda^{-2(n-r)/n} \longrightarrow_{\lambda \to \infty} 0\,. \quad \square$$

Section 9.5.C. (December, 2013)

By Theorem 9.5.2, the fields of definition of all weakly eutactic lattices up to dimension 5 are totally real. Batut (private communication) has made an exploration of all minimal classes lying below that of $\mathbb{A}_6$. He found eutactic lattices which are defined over non-totally real fields, notably on a cubic field with mixed signature.

(July, 2017). Weakly eutactic lattices belonging to Voronoi paths have totally real fields of definition. Actually such a minimal class has a parametrization by matrices $M(t) = A + t(B - A)$, $t \in [0,1]$. These matrices may be written in the form $M(t) = -t\,A\big(\frac{t-1}{t} I_n - A^{-1}B\big)$, which shows that if θ is a root of $p := \det(M(t))$ then $\frac{\theta-1}{\theta}$ is an eigenvalue of $A^{-1}B$ (and indeed $\theta \mapsto \frac{\theta-1}{\theta}$ puts these roots in one-to-one correspondence with the set of eigenvalues $\neq 1$ of $A^{-1}B$). Hence all roots of p, and consequently of its derivative p' are real, the field of definition of the weakly eutactic lattice (if any) is totally real. (Moreover, since p has no roots in $[0,1]$, we see that p' has at most one root in $(0,1)$, which gives a direct proof of the uniqueness of weakly weakly eutactic in minimal classes of perfection co-rank 1.)

The question of the field of definition also arises for dual-eutactic lattices. In dimension 4, the class a_8 contains both a eutactic lattice, defined on $\mathbb{Q}(\sqrt{3})$, and a dual-eutactic lattice, defined on the cubic field K of discriminant -244, a quadratic discriminant, so that K is (up to conjugacy) *the* cubic subfield of the Hibert class field of the quadratic field $\mathbb{Q}(\sqrt{-61})$ (which has class number $h = 6$); see Exercise 9.5.1.

Section 9.7.C.

The minimal classes in dimensions 6 and 7 have been classified in a work by Ph. Elbaz-Vincent, H. Gangl and C. Soulé on the K-theory of $\mathbb{Z}$. The data can be read in [E-G-S2]. There are 5 634 classes in dimension 6 and 10 722 899 in dimension 7, the classification of which has been completed some months after the 2005 Oberwolfach meeting. [In the notation of the tables of [E-G-S2], the perfection rank is $n + 1$.]

Minimal classes are identified by the $n \times n$ positive, definite matrix $S\,{}^tS$ (called the *Bacher matrix* by Anne-Marie Bergé and the *barycenter matrix* in [E-G-S2]): two systems of minimal vectors define the same class if and only if their Bacher matrices are equivalent over $\mathrm{Gl}_n(\mathbb{Z})$, that is, define isometric lattices.

Whereas Batut classified all cells containing a weakly eutactic lattice, using a specific *gradient algorithm* and using the Bacher matrix only for calculating the automorphism group of the class, the authors of [E-G-S2] restricted themselves to minimal classes, using all the strength of the Bacher matrix. (The classification of (weakly) eutactic lattices, though possible using their data, would need heavy calculations.)

From the Bacher matrix (or barycenter matrix) Bc of a class $\mathcal{C}$, one can easily see whether $\mathcal{C}$ contains a strongly (semi-) eutactic lattice (definition 3.2.C1). The recipe is given by the proposition below, the proof of which is straightforward from the definitions.

Proposition 9.7.C1. (A.-M. Bergé, J. Martinet).

Let $\mathcal{C}$ be a minimal class, let S be the set of minimal vectors of a lattice $\Lambda \in \mathcal{C}$, let $\mathrm{Bc} = S\,{}^tS$ be the corresponding Bacher matrix, and let S_1 be the set of minimal vectors of the form associated with Bc^{-1}.

(1) *$\mathcal{C}$ contains a strongly eutactic lattice if and only if $S_1 = S$. Then Λ is strongly eutactic, and Bc^{-1} is a Gram matrix for Λ.*

(2) *$\mathcal{C}$ contains a strongly semi-eutactic lattice if and only if S_1 is contained in S. Then Λ is strongly semi-eutactic, and S_1 is the set of of minimal vectors of Λ having non-zero eutaxy coefficients.*

Using this device, Elbaz-Vincent and Gangl were able during the 2005 Oberwolfach meeting to obtain the complete list of the 6-dimensional strongly (semi-) eutactic lattices: 21 strongly and 6 semi-strongly eutactic lattices. (Only 20 + 3 were previously known.) The file *strongeut.gp* of my homepage contains the complete classification of strongly eutactic or semi-eutactic lattices up to dimension 6 and examples in dimensions 7, 8, 9, and 10.

One of the main problems in this theory is:

What kind of information can be extracted from Bc*?*

On November 27., 2012, while I was delivering a talk in the algorithmic seminar in Bordeaux, Gabriel Nebe pointed out to me the existence of a paper by Plesken ([Pl3] in the complementary bibliography) in which the author describes an algorithm which, given Bc and s, outputs an $n \times s$ matrix S solving the equation $S\,{}^tS = \mathrm{Bc}$. On can thus go from Bc back to a minimal class. However one cannot obtain directly from Bc the perfection rank, nor forecast the nature of Λ with respect to (weak) eutaxy.

Also, *is there a simple interpretation of the standard invariants of* Bc *(minimum, kissing number, Smith invariant) in terms of the corresponding minimal class?*

The question of the connection which could exist between the Ash and the Bavard formulae has been solved in a 2007 joint work with Anne-Marie Bergé ([B-M8]). It is shown that if a minimal class $\mathcal{C}$ does not contain any weakly eutactic lattice, then $\inf_{\mathcal{C}} \gamma$ is attained at a

unique (up to similarity) lattice $L \in \partial\mathcal{C}$, that L is weakly eutactic, but not eutactic, and that Bavard's weighted sum with signs restricted to the set cells for which $\inf_{\mathcal{C}} \gamma$ is attained on L is zero.
[In the statement above, semi-eutactic lattices play no special rôle inside the set of all weakly eutactic lattices, in contrast to e.g. Corollary 3.8.C1.]

Section 11.6.C. (March, 2020)

Consider the matrices $M(t)$, P and the Moebius function φ below:

$$M(t) = \begin{pmatrix} 1 & -1+2t & -t \\ -1+2t & 1 & -t \\ -t & -t & 1 \end{pmatrix}, \ P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \ \text{ and } \varphi(t) = \tfrac{1-t}{1+t},$$

where $M(t)$, $\frac{1}{3} < t < \frac{1}{2}$, represents the dual-minimal class $\mathcal{C}$ with $s = s^* = 4$ of Section 9.2. Then the equalities

$${}^tPM(t)P = \tfrac{4t(1-t)}{1+t}\, M(\varphi(t)) \ \text{ and } \ {}^tP = P$$

show that $\mathcal{C}$ is globally isodual of symmetric type. The fixed point of φ ($t = -1 + \sqrt{2} = 0.414\dots$) defines the *ccc* lattice; see Exer. 6.3.1.

The same kind of result holds with the matrix displayed after Theorem 9.2.2 for the dual-minimal class with $s = s^* = 3$, taking $\varphi(t) = -\frac{t}{t+1}$ and $P = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$; see Exer. 9.2.2.

Section 13.1–3.C. (December, 2013)

In the usual Voronoi algorithm, because minimal classes of perfection co-rank 1 are necessarily well rounded, *dead ends* never occur. To get rid of them (at least in the equivariant algorithm), it suffices to restrict oneself to well rounded (= bounded) minimal classes: bounded minimal classes are the natural setting of the theory; see also Proposition 9.4.C1.

Section 13.3.C. (September, 2016)

In a previous edition of the file *erratang.tex* I wrote that one should suppress "$\in \mathbb{A}^{-1}\mathcal{T}\mathbb{A}^{-1}$". This is indeed correct.

In the case of the action of a group G the sum in this displayed formula is invariant by the transpose of G, not by G itself, except when $G = {}^tG$, e.g. in the case of the regular representation.

Section 14.4.C. (July, 2016)

We refer to Rogers's packing bounds to obtain reasonable upper bounds for the Hermite invariant beyond dimension $n = 8$. Three kinds of improvements have been obtained since the book was written: (1) In [Cn-El], Cohn and Elkies have improved Rogers's bounds in the range $4 \le n \le 36$. The inequalities they prove rely on the choice of a function on $\mathbb{R}^n$ having convenient properties together with its Fourier transform. In dimensions 8 and 24 they prove bounds very close to those which occur for $\mathbb{E}_8$ and Λ_{24}, respectively, and conjecture that these bounds should be attained for a well chosen function.

(2) In [Cn-Km1], Cohn and Kumar have proved that the Leech lattice is the unique critical lattice (up to scale) in dimension $n = 24$.

(3) Using modular forms Maryna Viazovska ([Viaz]) constructed a function which shows that the density of sphere packings in dimension 8 is bounded above by that of E_8; and the case of dimension 24 was then soon dealt with similarly ([C-K-V-al]; see also [Oe4]).

Independently of any application to lattice theory, it should be noticed that Hales (see [Hl1]) has found the exact bound for dimension 3 ("Kepler's conjecture"). As a consequence the density of any sphere packing in one of the dimensions $2, 3, 8$ and 24 is bounded above by the density of the densest known lattice packing.

Section 14.5.C. (Updated on June 14th, 2021, then February 14th, 2022.)

Thanks to Cohn–Kumar's [Cn-Km1]; see above, Section 14.4.C (resp. Poor–Yuen [P-Y4]; see above, Section 6.4.C) the results displayed in Table 14.5.1 for $n = 24$ (resp. for $n = 5, 6, 7$) are now known to be optimal, and attained exactly on the self-dual Leech lattice (resp. on the four extreme lattices $\mathbb{D}_5$, $\mathbb{A}_5^3$, $\mathbb{E}_6$, $\mathbb{E}_7$ and their four duals).

In the other dimensions the known upper bounds for γ'_n come from upper bounds for γ_n, via the inequality $\gamma'_n \leq \gamma_n$. This upper bound is strict if the dual of a critical lattice is not critical, a fact we *a priori* do not know. In this respect the case of dimension 9 is particular: using Theorem 2.8.7 (2) of [M] and the now known exact value of γ'_5, we can show the inequality $\gamma'_9 < 2\,(\leq \gamma_9)$.

Here are three remarks concerning the search of lattices Λ on which γ' takes high values. (Since ${\gamma'}^2$ is rational on rational lattices, we shall list values of ${\gamma'}^2$ rather than of γ', though we cannot exclude that γ'_n be an algebraic irrationality for some n.)

(1) The set $S(\Lambda)$ carries the structure of a spherical design of level some well-defined odd integer ℓ (if $n \geq 2$; if $n = 1$, $\ell = \infty$); conjecturally, $n = 1$, 3, 5, 7 or 11. Lattices with $\ell \geq 5$ were called *strongly perfect* by Venkov. In particular he proved that if $\ell \geq 5$ (resp. $\ell \geq 7$, then we have

$$\gamma'(\Lambda)^2 \geq \tfrac{n+2}{3} \text{ (resp. } \gamma'(\Lambda)^2 > \tfrac{n+2}{3}\text{)}\,.$$

This "Venkov bound" $\frac{n+2}{3}$ is an interesting value to give an idea of what can be expected for ${\gamma'}^2$.

(2) By contrast with the Hermite invariant γ, when searching for high values of γ' on sections of a lattice, the densest sections are often not the best choices.

(3) It often happens that for a lattice Λ having an odd minimum m, Λ and its even sublattice Λ_{even} have the same dual. Then passing from Λ to Λ_{even} improves γ' by a factor at least $\frac{m+1}{m}$, generally, $\frac{4}{3}$ if $m = 3$; $\frac{8}{5}$ for a lattice generated by vectors of norm 5 with paiwise scalar poducts ± 5 or ± 1, as in the example with $n = 13$ and $m = 5$, where Λ_{even} is Conway-Sloane's example in [S-S9]. After rescaling such lattices produce integral lattices of minimum 4.

On a lattice Λ which is not self-dual, Λ and Λ^* are non-similar lattices on which γ' has the same value. We shall generally quote only one of them. Self-dual lattices will be t-modular for some t (because we shall meet only rational lattices). Denoting by m the minimum of Λ, we then have $\gamma'(\Lambda)^2 = \frac{m^2}{t}$.

Next we consider the results of Table 14.5.1 in the range 9—23.

Our largest known value of γ' occurs on several lattices when $n = 9$, 10, 11, 14, 17, 19, 20. If some of them is not dual-extreme, *the given lower bound is strict*. This is the case exactly for $n = 17$ and $n = 19$: for $n = 17$ (${\gamma'}^2 = 6$), on even sublattices of lattices of minimum 3 or 5; for $n = 19$ (${\gamma'}^2 = \frac{48}{7} = 6,857...$), on even sublattices of lattices of minimum 3, or on cross-sections of the "Kleinian", 7-modular lattice of dimension 20, with ${\gamma'}^2 = \frac{64}{7} = 9.142....$

Modular lattices occur with $(m, t) = (4, 4)$ ($n = 10$, two lattices), $(4, 3)$ ($n = 12, 14$), $(4, 2)$ ($n = 16$), and $(8, 7)$ ($n = 20$, three lattices).

Strongly perfect lattices give examples for $n = 10$, 12, 14, 16, 18, 20, 21 and 23 (and $2, 4, 8, 24$).

We now illustrates the remarks (2) and (3) above. The densest 2-dimensional sections of $O_{23}^* \simeq O_{23}$ are represented by matrices $\left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right)$, defining by orthogonality the lattice O_{21} and a new lattice O'_{21}, respectively. Taking the successive densest hyperplane sections we obtain a well-defined descending series O'_{21}, ..., O'_{16} distinct from O_{21}, ..., O_{16}, the next element of which is isometric to O_{15}.

By orthogonlity to any vector of the fourth level of ${\mathrm{O}'}_{20}^*$ (resp. to a conveniently chosen vector of the third level of ${\mathrm{O}'}_{18}^*$), we obtain a lattice L'_{19odd} (resp. L'_{17odd}). Denote by L'_{19} and L'_{17} their respective even parts (L'_{17} is isometric to one of the Plesken-Pohst lattices for minimum 4). We have

$$\min L'_{19} = 4\,,\ \min {L'_{19}}^* = \frac{12}{7}\,,\quad \min L'_{17} = 4\,,\ \min {L'_{17}}^* = \frac{3}{2}\,.$$

It turns out that both the sets of minimal vectors of their duals are of rank 12, strictly smaller than their dimensions, so that these lattices (indeed, extreme) *are not dual-extreme*. We thus have the strict inequalities

$$\gamma'_{19} > \tfrac{48}{7} = 6.857... \quad \text{and} \quad \gamma'_{17} > 6\,.$$

We now list lattices having the largest known value of γ'_n, skipping dimensions 13, 17, and 19 yet considered above.

In some dimensions we know a unique lattice to within duality, having a rather large γ' invariant, that can be conjectured to be the unique dual-critical lattice in its dimension. These are the modular lattices K_{12} and Λ_{16}, and pairs (L, L^*) with $L = K'_{18}$, K'_{21}, Λ_{22} and Λ_{23}.

We are now left with dimensions 9, 10, 11 and 14. The lower bound given for $n = 9$ ($\gamma'^2 \geq \frac{16}{5}$) is attained on the Coxeter lattice $\mathbb{A}_9^{(2)}$ and on the Baril lattice described in Section 3.8.C above of this complement (and their duals).

In dimension 10 we have $\gamma'^2 \geq 4$, value attained on four lattices mentioned above: the 4-modular lattices $\sqrt{2}\mathbb{D}_{10}^+$ and Souvignier's Q_{10} (isometric to the even sublattice of $\bigwedge^2 \mathbb{D}_5$); a section of $\Lambda_{11}^{\mathrm{mid}}$), and the pair of strongly perfect lattices $(K'_{10}, {K'}^*_{10})$.

In dimension 11, we also have $\gamma'^2 \geq 4$, value attained on three pairs (L, L^*) of non-isodual lattices, with representatives K_{11}, Coxeter's $\mathbb{A}_{11}^3$ and $\Lambda_{11}^{\mathrm{mid}}$, the orthogonal to $\Lambda_{13}^{\mathrm{mid}}$ in the Leech lattice (*not* a laminated lattice).

In dimension 14, the value $\gamma'^2 = \frac{16}{3}$ is attained on the 3-modular, strongly perfect lattice Q_{14}, but also on Λ_{14}, This extreme lattice, which has a non-eutactic dual, is nevertheless dual-extreme, as checked by direct calculation.

Questions.

(1) Are the values of γ'^2 strictly smaller than $\frac{n+2}{3}$ (the Venkov bound for 5-sesigns) in dimensions 9, 11, 13, 17 and 19 ?

(2) Is the lower bound ${\gamma'_n}^2 \geq 4$ exact in dimensions 10 and 11 ?

I expect the answers to be positive, with a modicum of doubt for $n = 17$, where the known lower bound is close to Venkov's.

Beyond dimension 24, good examples are sparse, principally derived from modular lattices. This applies in particular for
$n = 26, (m, \ell) = (6, 3), \gamma'^2 = 12\,$; $n = 32, (m, \ell) = (6, 2), \gamma'^2 = 18\,$;
$n = 48, (m, \ell) = (6, 1), \gamma'^2 = 36\,$; $n = 72, (m, \ell) = (8, 1), \gamma'^2 = 64\,$.
Other reasonably good examples are obtained using cross-sections of

the previous lattices, for instance, we have the lower bound ${\gamma'_{25}}^2 \geq 9$. Also even sublattices of some unimodular lattices of minimum 3, produce examples with ${\gamma'}^2 = 11$ for $n = 27$ and ${\gamma'}^2 = 12$ for $n = 28, 29, 31$. However, Nebe's extremal 5-modular lattice of dimension 28 has ${\gamma'}^2 = \frac{64}{5} = 12.8$, and Bachoc's 2-modular, 32-dimensional lattice over the Hurwitz quaternions has sections of dimension 31 with ${\gamma'}^2 = \frac{27}{2} = 13.5\,$.

SECTION 16.2.C.

G. Nebe and B. Venkov have proved in [Ne-V3] that the Coxeter-Todd lattice K_{12} is the unique strongly perfect lattice in dimension 12. They proved in [Ne-V5] that the isodual lattice Q_{14} is the only 14-dimensional lattice L such that both L and L^* are strongly perfect; this result has been extended to dimensions 13 and 15 in [Ne-No-V]; see also [Nos1] for some complements, and to dimension 16 by Hu and Nebe ([Hu-Ne1],[Hu-Ne2]).
[One conjectures that the hypothesis on the dual lattice is not necessary.]

In [Hu-Ne1], the authors construct two strongly perfect "companion lattices" to the Barnes-Wall lattices in dimensions $n = 2^{2m} \geq 16$. In particular, the 16-dimensional examples constitute the first enlargement of the list in dimensions ≤ 23 established circa 2000 by Batut and Venkov.
Question Are these lattices 7-designs, or only 5-designs ?

G. Nebe and B. Venkov also classified (in [Ne-V4]) lattices whose set of minimal vectors is a higher design up to dimension $n = 24$, except possibly for $n = 23$ — but the appearance of other 6-designs in dimension 23 is very unlikely. This completes [Mar7], devoted to integral lattices of minimum $m \leq 5$. [**Warning**: in the statement of the classification theorem of [Mar7], the lattice O_{23} has been forgotten.]

SECTION 16.3.C.

C. Bachoc ([Bac4]) has proved using group theoretical arguments that all layers of the Barnes-Wall lattices are 7-designs in all dimensions $n \geq 8$.

On May 3rd, 2018, in the arXiv paper [Hu-Ne], Hu and Nebe have announced the construction of new infinite series of 7-designs sandwiched into the Barnes-Wall lattices and their duals in dimensions 16, 64, 256,... In particular this gives in dimension 16 new strongly perfect lattices, the first discovery in dimensions 1–23 since the list constructed in 1999 by Batut and Venkov for [Ven3].

SECTION 16.4.C.

For even unimodular lattices Λ the corrected formula in Theorem 16.4.1 which bounds the minimum (see the erratum), namely $\min \Lambda \leq 2+2\,\lfloor \frac{(\ell+1)\,n}{48} \rfloor$, reads $\min \Lambda \leq 2+2\,\lfloor \frac{n}{24} \rfloor$. It has been proved by Rains and Sloane ([R-S]) that this bound also holds for odd unimodular lattices except for the lattice O_{23}, of minimum 3; and Gaulter ([Ga]) has shown that for dimensions $n \equiv 0 \mod 24$, this bound is strict on odd unimodular lattices.

The last sentence (... $n = 80$ with $\ell = 1$) refers to [Bac-Ne2] where the authors construct two unimodular lattices of minimum 8. At this date (1998), extremal even unimodular extremal lattices were known to exist in dimensions $n = 8k$ up to $n = 64$ and in dimension 80. On August 11th, 2010, Gabriele Nebe announced that she had constructed an even unimodular lattice of dimension 72 and minimum 8, now published as [Ne6].

At the date of December 13th, 2013, four even unimodular lattices are known in dimension 48. Two of them are quoted in [C-S], and the other two have been found by Nebe; see [Ne5], [Ne8], [Ne9]. Also four lattices are now known in dimension 80; see [Ste-Watk] and [Watk].
[(Unpublished.) Using the four known even, extremal, 2-modular lattices of dimension 32 and minimum 6, I have constructed four probably distinct extremal even unimodular lattices in dimension 64. The construction, an analogue of a construction of Barnes-Wall lattices, consists in doubling the dimension, obtaining alternatively 1- and 2-modular lattices, but not extremal beyond dimension 64.]

.../...

More on Craig's lattices (appendix to Section 5.4)

Table 5.4.12 was calculated using the *PARI* system. We reproduce it below, then list the s invariants of some more Craig lattices of minimum $2r$ with $r \le \lfloor \frac{p+1}{4} \rfloor$, or their norm N when it exceeds $2r$.

Table 5.4.12. Lattices $\mathbb{A}_{p-1}^{(r)}$ with $29 \le p \le 47$ and $3 \le r \le (p+1)/4$.

	3	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
29	6496	7917	5684	1421	580					
31	8835	11625	7905	5735	930	465				
37	19092	31968	29304	14430	13320	1665	888			
41	29520	57605	61008	36490	31980	4920	$N = 20$	1066		
43	36421	74046	86688	63812	25929	28896	2709	$N = 22$	903	
47	52969	122153	154583	141611	63779	69184	4324	$N = 22$	$N = 24$	1081

Complement 1. Beyond (p+1)/4.
$\underline{p = 29}$. $r = 8 : N = 20, s = 3248$.
$\underline{p = 37}$. $r = 10 : N = 24, s = 777$.
$\underline{p = 41}$. $r = 11 : N = 28, s = 2460$.
$\underline{p = 43}$. $r = 12 : N = 28, s = 129$.
$\underline{p = 47}$. $r = 13 : N = 34, s = 4324$.
$\underline{p = 53}$. $r = 14 : N = 36, s = 4134$.

Complement 2. Exceptional pairs (p,r). For $p \le 53$ and $r \le (p+1)/4$, the only exceptions to the rule $N(\mathbb{A}_{p-1}^{(r)}) = 2r$ are the following ones, for which $N(\mathbb{A}_{p-1}^{(r)}) = 2r + 2$:
$p = 41$, $r = 9$, $N = 20$, $s = 10086$; $p = 43$, $r = 10$, $N = 22$, $s = 3612$;
$p = 47$, $r = 10$, $N = 22$, $s = 12972$; $p = 47$, $r = 11$, $N = 24$, $s = 3243$;
$p = 53$, $r = 11$, $N = 24$, $s = 12402$; $p = 53$, $r = 12$, $N = 26$, $s = 1696$.

Complement 3. Larger values of p.
$\underline{p = 59}$: $r = 3$, $N = 6$, $s = 138591$;
$r = 4$, $N = 8$, $s = 424328$;
$r = 5$, $N = 10$, $s = 759684$.

$\underline{p = 61}$: $r = 3$, $N = 6$, $s = 159820$;
$r = 4$, $N = 8$, $s = 504165$;
$r = 5$, $N = 10$, $s = 924516$.