Last update: April 3rd, 2014

ERRATUM AND COMPLEMENTS TO MONOGRAPHIE 37 RÉSEAUX EUCLIDIENS, DESIGNS SPHÉRIQUES ET FORMES MODULAIRES

JACQUES MARTINET

ABSTRACT. We present here errata and complements for some of the papers published in "Monographie 37 de l'Enseignement Mathématique". The papers in which I am not involved will be considered only if the authors wish to send me their remarks.

N.B. Recent references are those of "Corrected and extended reference list of the book "Perfect Lattices in Euclidean Spaces"." (This homepage.)

Summary of the book.

(1) *B.Venkov* (Notes by *J. Martinet*),
Réseaux et designs sphériques.
(2) *C. Bachoc, B.Venkov* (with an appendix with *G. Nebe*),
Modular forms, lattices and spherical designs.
(3) *J. Martinet, B.Venkov* (with an appendix by *R. Coulangeon*),
Les réseaux fortement eutactiques.
(4) *J. Martinet*,
Sur certains designs sphériques liés à des réseaux entiers.
(5) *R. Coulangeon*,
Voronoï theory over algebraic number fields.
(6) *J. Martinet* (with an appendix by *C. Batut*),
Sur l'indice d'un sous-réseau.
(7) *R. Bacher, B.Venkov*, Réseaux entiers unimodulaires sans racines en dimensions 26 et 27.
(8) *P. Engel, L. Michel* †, *M. Senechal*, New geometric invariants for Euclidean lattices.

1. First paper.

Updated bibliography.

The paper [L-S-T] has appeared:
[L-S-T] W. Lempken, B. Schröder, P.H. Tiep, *Symmetric squares, spherical designs, and lattice minima. With an appendix by Christine Bachoc and Tiep*, J. Algebra **240** (2001), 185–208.

An English updated version of the book [M] has appeared:
[M'] J. Martinet, *Perfect Lattices in Euclidean Spaces*, Grundlehren **327**, Springer-Verlag, Heidelberg (2003).

The paper [R-S] has appeared:
[R-S] E. Rains, N.J.A. Sloane, *The shadow theory of modular and unimodular lattices*, J. Number Theory **73** (1998), 359–389.

The paper [N-R-S] has appeared:
[N-R-S] G. Nebe, E. Rains, N.J.A. Sloane, *The invariants of the Clifford groups*, Des. Codes Cryptogr. **24** (2001).
See also
[N-R-S2] G. Nebe, E. Rains and N. Sloane, *Codes and invariant theory*, Mathematische Nachrichten **274-275** (2004), 104-116,
and also the book by the same authors:
Self-Dual Codes and Invariant Theory, Algorithms and Computation in Mathematics nu. 17, Springer-Verlag, Heidelberg, 2006.

Complements.

Section 18. Using the results of [N-R-S], C. Bachoc (*Designs, groups, and Lattices*, J. Th. Nombres de Bordeaux **17** (2005), 25–44, issue dedicated to the *Journées Arithmétiques* of Graz, 2003) has proved that from dimension 8 onwards, the automorphism groups of the Barnes-Wall lattices have no non-trivial invariants in degrees $d \le 6$. As a consequence, all non-zero orbits of these automorphism groups are spherical 7-designs. In particular, all layers of the Barnes-Wall lattices are spherical 7-designs. [In the language of Section 5, the fundamental degrees for $\mathrm{Aut}(\mathrm{BW}_n)$ ($n = 2^p \ge 8$) are equal to 2 or ≥ 7.]

Section 19. The list of strongly perfect lattices displayed in Tables 19.1 and 19.2 was constructed by Batut and Venkov. No new strongly perfect lattices have been discovered since in dimensions $n \le 25$. In dimension 26, three strongly perfect lattices are known, namely one 3-modular extremal lattice, the integral lattice of minimum 4 and determinant 3 discovered by G. Nebe, and its dual.
G. Nebe and B. Venkov (*Low dimensional strongly perfect lattices. I: The* 12*-dimensional case*, L'Enseignement Mathématiques **51** (2005), 129–163) have recently proved that the Coxeter-Todd lattice K_{12} is the only strongly perfect 12-dimensional lattice.

They have also proved an analogous result in dimension 14, but under an extra condition, and work with E. Nossek has extended this last result to dimensions 13 and 15; see the complements to *Perfect Lattices in Euclidean Spaces*, Section 16.3 C (ref. [Ne-No-V]).

More on group theory. C. Bachoc's written talk of the "Journées Arithmétiques of Graz, 2003" contains the list of all known strongly perfect lattices in dimensions $n \le 26$ whose automorhism groups have fundamental non-trivial degrees ≥ 5.

2. Third paper.

Erratum.
In Table 8.1, a lattice with $n = 5$, $r = s = 12$ and minimum 10 has been forgotten; see nu. 5 in Martinet's catalogue of perfect lattices.

Updated bibliography.

[Bt] C. Batut, *Classification of quintic eutactic forms*, Math. Comp. **70** (2001), 395–417.

Complements.
One more strongly eutactic 6-dimensional lattice has been found with $s = r = 15$ and minimum 8 (A.-M. Bergé, J. Martinet, *Symmetric Groups and Lattices*, Monatshefte Math. **140** (2003), 179–195, the lattice with Gram matrix C_6 displayed after Theorem 4.3). More recently, Elbaz-Vincent/Gangl/Soulé have obtained the complete classification of 6-dimensional cells, and consequently, that of strongly eutactic 6-dimensional lattices. The results for dimension 7 could be extracted from their work. See the file devoted to strongly eutactic lattices in Martinet's home page, and P. Elbaz-Vincent, H. Gangl, C. Soulé, *Perfect forms and the cohomology of modular groups*, [E-G-S]; preprint at arXiv:math/1001.0789v1.

3. Fourth paper.

Erratum.
In the theorem stated in the introduction (page 137 of the book), the lattice O_{23} has been forgotten. The correct statement reads as follows:

Theorem. *The integral primitive lattices* Λ *of minimum* $m \le 5$ *whose set* $S(\Lambda)$ *of minimal vectors is a spherical* 7*-design are* $\mathbb{Z}$*, the root lattice* $\mathbb{E}_8$*, the shorter Leech lattice* O_{23}*, the three laminated lattices* Λ_{16} *(the Barnes-Wall lattice* BW_{16}*),* Λ_{23} *and* Λ_{24} *(the Leech lattice), and the even unimodular lattices of dimension* 32 *and minimum* 4 *(which have not been classified). In particular, minimum* 5 *is not possible.*

Complements.
The classification of lattices whose sets of minimal vectors constitute a 6-design (or 7 -design) has been solved in dimensions $n \le 24$ (except for an open case if $n = 23$) by Nebe and Venkov in *On lattices whose minimal vectors form a* 6*-design*, preprint (2006); European J. Combin. **30** (2009), 716–724. (Special issue dedicated to Eiichi Bannai's 60th birthday.)

4. Sixth paper.

Erratum.
p. 173, Example 3.3. In the three formulae involving e, replace the denominators 2 by d.

In Section 9, case d=6.
line +2 (p. 192, l. -5 in the book): read i.e., $m_3 \le 2$.
(6,0,2) (page 193, line 8 in the book), after "donc $e_6 \cdot f' = 0$", read:
$\forall\, i,\ e_i \cdot f = 1 \Longrightarrow N(f) = 2$ and $N(f - e_i - e_6) = 2\, e_i \cdot e_j \ge 1$
$\Longrightarrow N(f) \ge \frac{1+5/2}{3} = \frac{7}{6} < 2$, a contradiction; see also Proposition 2.2 a below.

In Section 9, case d=9, (page 194, line 3 in the book), read: i.e., $m_3 \le 2$.

In Section 9, case (2,4,2), (page 195, line -5 in the book), in "$e =$", read $-e_3$.

In Section 9, case (4,3,1) (page 196, line 12 in the book), read ... $+ e_4 + e_8$ instead of ... $+ e_4 e + e_8$.

pp. 198–199. Several slips in the long proof showing the impossibility of $(3, 2, 2, 1)$ have been found by Achill Schürmann. The proof is otherwise correct, but can now be skipped since all cyclic quotients in dimensions up to 9 have been classified in [K-M-S].

In Section 10, page 200, line -6, read $\frac{1}{3} \sum_{i=1}^{8} a_i e_i$ instead of $\frac{1}{3} \sum_{i=0}^{8} a_i e_i$.

p.204, line 18. Actually, all scalar products $e_i \cdot e_j$, $3 \le i < j \le 6$ are also equal to y, so that $N(e) = \frac{10+12y}{16}$. The inequality $N(e) \ge 1$ now reads $y \ge \frac{1}{2}$, whence $y = \frac{1}{2}$, which implies $N(e - e_3) = \frac{3}{4} < 1$, a contradiction.

Updated bibliography.

One can replace [M] by the better reference

[M'] J. Martinet, *Perfect Lattices in Euclidean Spaces*, Grundlehren **327**, Springer-Verlag, Heidelberg, 2003,

and also [M2] by [M'], Section 6.4.

One should also add to [Ry] the reference

[B-R] Baranovskii, S.S. Ryshkov, *Derivation of perfect lattices from admissible centerings*, Russian Math. Survey **40, 4** (1985), 155–156. (Russian original: 1985.)

as well as the recent

[K-M-S] W. Keller, J. Martinet, A. Schürmann, *On classifying Minkowskian sublattices*, Math. Comp. **81** (2012), 1063-1092; preprint at arXiv:0904.3110v3.

Maxim Anzin, in an e-mail dated March 23rd, 2004, pointed out to me that the three possible structures which were forgotten in [Zah] were corrected by the author in a preprint written under the name of N. V. Novikova, a preprint (in Russian) that I have never seen.

Also, the name Zahareva may occur in print as Zaharova or Zakharova.

Complements.

Section 2. In a joint work with Anne-Marie Bergé (*On Perfection Relations in Lattices*, preprint, arXiv: math.NT/0611220 (8 Nov. 2006), 26 pp.); Contemporary Math. 493 (2009), 29–49, we proved the following complement to Watson's Theorem 2.2:

Proposition 2.2 a. *If equality holds in Theorem 2.2, we have $a_i \le \frac{d}{2}$ for all i, and if $d \ge 4$, equality holds for at most one index i.*

This simplifies some proofs, and in particular immediately shows that the system $(6, 0, 2)$ is impossible.

Whenever Watson's condition holds, there is a unique perfection relation relating the orthogonal projections to the vectors e_i and e'_i, namely

$$\sum_{i=1}^n a_i\, p_{e_i} = \sum_{i=1}^n a_i\, p_{e'_i}\,.$$

This is quoted in Appendix 3 to the joint paper with A.-M. Bergé *A generalization of some lattices of Coxeter*, Mathematika **51** (2004), 49–61.

Section 7. For cyclic quotients with order $d \le 6$), all lattices Λ constructed as $\Lambda = \langle e_1, \dots, e_n, e = \frac{a_1\, e_1 + \cdots + a_n\, e_n}{d} \rangle$ with $e_1, \dots, e_n \in S(\Lambda)$ and a_i not divisible by d exist provided Watson's inequalities hold for all divisors $d' > 1$ of d and moreover have for $s_{\min}$ and r the value predicted by Watson's identities, except for a short list of exceptions corresponding to small dimensions n. Here are the results, which were only partially given in the paper.

- $d = 2$, $n \ge 4$: $s = 12$ and $r = 10$ if $n = 4$, $s = r = n$ if $n \ge 5$.
- $d = 3$, $n \ge 6$: $s = 12$ and $r = 11$ if $n = 6$, $s = r = n$ if $n \ge 7$.
- $d = 4$, $n \ge 7$, $m_1 \ge 4$, $m_2 \ne 0$ if $n = 7$. Except if $(m_1, m_2) = (4, 3)$, $(6, 1)$ or $(8, 0)$, we have $s = n + 8$ and $r = n + 6$ if $m_1 = 4$, and $s = r = n$ if $m_1 \ge 5$.
- $d = 5$, $n \ge 8$ (and $m_1 \ge m_2$): $s = 2n$ and $r = 2n - 1$ if $(m_1, m_2) = (4, 4)$, $(6, 2)$, $(8, 1)$ or $(10, 0)$, and $s = r = n$ otherwise.

• $d = 6$, $n \geq 9$. All systems (m_1, m_2, m_3) satisfying Watson's conditions

$$m_1 + m_3 \geq 4\,,\ \ m_1 + m_2 \geq 6 \ \text{ and } \ m_1 + 2m_2 + 3m_3 \geq 12$$

exist and have the predicted values for $s_{\min}$ and r except $(4,5,0)$ and $(5,1,3)$ for which $s_{\min} = 23$ (by repeated application of Watson's identities).

Section 9, Proposition 9.1. Zahareva's identity holds true for any system (m_1, m_2), with

$$(m_1 - 4) \sum_{m_1+1 \leq i \leq m_2} (N(e' - e_i) - N(e_i)) + (m_2 - 4) \sum_{1 \leq i \leq m_1} (N(e - e_i) - N(e_i))$$

in the right hand side. When $m_1 = m_2 = 4$, there is a unique perfection relation relating the orthogonal projections to the vectors e_i and $e'_i = e - e_i$ or $e' - e_i$, namely $\sum_{i=1}^n p_{e_i} = \sum_{i=1}^n p_{e'_i}$.

There is a similar result for denominator 7, involving m_1, m_2, m_3, and assuming that $m_1 = m_2 = m_3 = 3$ for the perfection relation. This accounts for the values $s = 18$, $r = 17$ in Remark 9.2.

Section 9, Remarque 9.2, page 200 in the book (November 24th, 2008).

The complete classification for dimension 9 has been obtained in the joint work [K-M-S] with Wolfgang Keller and Achill Schürmann (*On classifying Minkowskian sublattices*; preprint at arXiv: 0904.3110v1). The new structures for quotients Λ/Λ' which occur are (7), (8), (9), (10), (12), $(6,2)$ and $(4,2,2)$, which all exist in the lattice Λ_9, and $(4,4)$, which exists only in a perfect, strongly eutactic lattice with $s = 81$.

The proofs make intensive use of linear programming.

Sections 9 and 10 (*July 3rd, 2006*).

The bound $[\Lambda : \Lambda'] \leq 9$ for 8-dimensional lattices (except for $\mathbb{E}_8$, for which an elementary quotient of order 16 exists) was stated without a proof by Watson. For the sake of completeness, we present a detailed, handy-computational proof as an appendix.

5. Eighth paper.

Updated bibliography.

[EMS01] Marjorie Senechal's home page
http://www.math.smith.edu/~senechal/books.html refers to a 284 pages, 2003 preprint, available as an I.H.E.S. 2004 preprint at http://www.ihes.fr/ .

Appendix: complements for dimension 8

In dealing with dimension 8 (Sections 9 and 10), we have left aside the possible large indices: all the details have been written for $\imath \leq 9$, and we just trusted Watson's statement without proofs for $\imath > 9$. Here we shall provide proofs. Note that if one makes use of the results on Hermite's constant in dimension 8, there just remains to consider the range $10 \leq \imath \leq 15$, since $\gamma_8^4 = 16$ and is attained only on lattices similar to $\mathbb{E}_8$. We must thus consider for Λ/Λ' the structures (10), (11), (12), $(6,2)$, (13), and (15), (note that (14) is impossible because index 7 does not exist in dimension 8).
[Since γ_8 and the corresponding critical lattices can be easily classified using the corresponding results in dimensions 6 and 7 (a short proof due to A.-M. Bergé is written in [M], Theorem 6.6.1), it is reasonable not to bother with larger indices. However, using only the value of g_5 and the bound for g_8 obtained by repeated use of Mordell's inequality (according to Watson's philosophy), one would have to consider the larger range $10 \leq \imath \leq 18$. Again, $\imath = 18$ is impossible (because $\imath = 9$ exists only for $\mathbb{E}_8$) and quotients Λ/Λ' of type (16) and $(8,2)$ are excluded (because denominator 8 does not exist in dimension 8). We are then left with the extra structures $(4,4)$, $(4,2,2)$, and (17).

We first consider the case when Λ/Λ' is cyclic, i.e.

$$\Lambda = \langle \Lambda', e = \frac{a_1 e_1 + \cdots + a_n e_n}{d} \rangle$$

for some d in the range $10 \leq d \leq 15$ (or $d = 17$) $1 \leq a_i \leq d' = \lfloor \frac{d}{2} \rfloor$. We denote by m_i the number of $j \in \{1, \ldots, 8\}$ such that $a_j = i$; we have $m_1 + \cdots + m_{d'} = 8$, and set $\sigma_1 = m_1 + 2m_2 + \cdots + d' m_{d'}$ and for a prime to d, we denote by σ_a the transform of σ_1 resulting from the transformation $e \mapsto a\,e$. Watson's inequalities then read $\sigma_a \geq 2d$. If equality holds for, say, σ_1, all vectors $e - e_i$ are minimal, so that if $m_i \neq 0$, then the denominator $d - i$ is possible in dimension 8. If $d = 10$, we have $m_5 = 0$ (because index 5 does not show up in dimension 7), m_1 and m_3 cannot be both zero, and a denominator $10 - 1 = 9$ or $10 - 3 = 7$ shows up, a contradiction. When it is proved that index 10 is impossible, we may again apply the same argument with $d = 11$, which will show inductively that the inequality must be strict in Watson's inequality for all $d \geq 10$.

Before looking at the possible values of d, we write down two general inequalities.

First, Watson's bound implies that $m_1 + (8 - m_1)d' \geq 2d + 1$, i.e. $m_1 \leq \frac{4d-6}{d-3} < 5$ for $d \geq 11$ odd (and similarly $m_1 \leq 4$ holds for $d \geq 10$ even).

Next, if d is odd, adding Watson's inequalities for e and $2e \mod \Lambda'$, we can sharpen $m_1 \leq 4$ to $m_1 \leq 3$ for $d \geq 11$ odd: for k even (resp. odd) the coefficient of m_k is bounded from above by $k + \frac{k}{2} \leq \frac{3(d-1)}{4}$ (resp. by $k + \frac{d-k}{2} = \frac{d+k}{2} \leq \frac{3d-1}{4}$), so that $m_1 + \frac{3d-1}{4}(8 - m_1) \geq 2(2d + 1)$, i.e. $m_1 \leq \frac{8(d-2)}{3d-13} < 4$ for $d \geq 11$. [If Watson's equality could hold, the bound $m_1 \leq 3$ would be correct only for $d \geq 13$.]

Using the action of $(\mathbb{Z}/d\mathbb{Z})^\times/\{\pm 1\}$, we see that the inequalities above hold for all m_k with k prime to d. In particular, when d is a prime, $m_k \leq 3$ holds for all k.

We now consider the cyclic quotients Λ/Λ' of order $d = 10$ to 15.

$\underline{d = 10}$. Recall that $m_5 = 0$, and consider the two Watson inequalities

$$\sigma_1 = m_1 + 2m_2 + 3m_3 + 4m_4 \geq 21 \quad \text{and} \quad \sigma_3 = 3m_1 + 4m_2 + m_3 + 2m_4 \geq 21$$

and their averaging $2(m_1+m_3)+3(m_2+m_4)\geq 21$. We have $m_5=0$ because index 5 is not possible in dimension 7, hence $2(m_1+m_3)+3(8-(m_1+m_3))\geq 21$, i.e. $m_1+m_3\leq 3$, which contradicts the lower bound $m_1+m_3\geq 4$ given by $5e$.

$\underline{d=11}$. We have five inequalities, obtained by performing permutations on $\sigma_1=m_1+2m_2+3m_3+4m_4+5m_5\geq 23$, $\sigma_2=2m_1+4m_2+5m_3+3m_4+m_5\geq 23$, $\sigma_3=3m_1+\cdots\geq 23$, ... Adding the first two inequalities above, we obtain $(*)$ $3m_1+6m_2+8m_3+7m_4+5m_5\geq 46$. We have $m_i\leq 3$. If $\max m_i=3$, we may assume that $m_1=3$, but then the maximal value of $\sigma_1+\sigma_2$ is $41<46$ (attained on $(3,0,2,2,1)$). Hence $\max m_i=2$, so that the system (m_i) is a permutation of $2^4,0)$ or $(2^3,1^2)$. In the first case, we may assume that $m_3=0$, which implies $\sigma_1+\sigma_2\leq 44<46$. In the second case, we may assume that the value 1 occurs for m_3 but not for m_1, which implies $\sigma_1+\sigma_2\leq 46$, with equality only on $(2,1,1,2,2)$ (and then, $\sigma_2=21<23$) or $(2,2,1,2,1)$ (and then, $\sigma_1=22<23$).

$\underline{d=12}$. Considering denominators 6 and 4, we see that $m_6=0$ and $m_4\leq 1$. Averaging the two inequalities $\sigma_1=m_1+2m_2+3m_3+4m_4+5m_5\geq 25$ and $\sigma_5=5m_1+2m_2+3m_3+4m_4+m_5\geq 25$, we obtain $\sigma=\frac{\sigma_1+\sigma_5}{2}=3(m_1+m_3+m_5)+2m_2+4m_4\geq 25$. Setting $t=m_1+m_3+m_5$, we have $\sigma=3t+2m_2+4m_4\leq 3t+2(7-t)+4=t+18$, and $\sigma\geq 25$ holds only if $t=7$ and $m_4=1$, whence $m_2=0$. But this contradicts $\sigma_4=m_1+2m_2+m_3+m_5\geq 8$.

$\underline{d=13}$. The proof roughly follows the one we gave for $d=11$. We have $\sigma=\sigma_1+s_2+s_3=6m_1+12m_2+13m_3+10m_4+10m_5+12m_6\geq 27\times 3=81$. If $\max m_i=3$, we may assume that $m_1=3$, so that $\sigma\leq 6\times 3+13\times 3+12\times 2=81$, value attained uniquely on the systems $(3,0,3,0,0,2)$, $(3,1,3,0,0,1)$ and $(3,2,3,0,0,0)$, for which $\sigma_1\leq 24<27$. There remains to consider systems which are a permutation of $(2^4,0^2)$, $(2^3,1^2,0)$ or $(2^2,1^4)$.

- In the first case, we may assume that $m_3=0$ and $m_1=2$, whence $\sigma\leq(6+12+12+10)\times 2=80<81$.
- In the second case, we may assume that $m_3=0\text{ or }1$ and $m_1=2$. If $m_3=0$, then $\sigma\leq(6+12+12)\times 2+10=80$. Let now $m_3=1$ (and $m_1=3$). If $m_2=0$, then $\sigma\leq(6+10+12)\times 2+(13+10)=79$; If $m_2=1$, then $\sigma\leq 81$ with equality only on the systems $(2,1,1,0,2,2)$ and $(2,1,1,2,0,2)$, for which $\sigma_2\leq 22$; finally, if $m_2=2$, then $\sigma_1\leq(1+2+6)\times 2+(3+5)=26$.
- In the third case, we may assume that $m_1=2$ and $m_3=1$, which implies $\sigma\leq(6+12)\times 2+(13+12+10+10)=81$, with equality only if $m_2=2$, and then $\sigma_1=26$, or if $m_6=2$, and then $\sigma_2=26$.

$\underline{d=15}$. We have $m_5=0$ and $m_2+m_6\leq 2$. Adding $\sigma_1,\sigma_2,s_4,s_7$, we obtain the inequality $14(m_1+m_2+m_4+m_7)+18(m_3+m_6)\geq 124$, whose L H S is bounded from above by $14\times(8-6)+18\times 2=120$.

[$\underline{d=17}$. This superfluous case can be dealt using the methods used for $d=13$.]

We now turn to non-cyclic quotients Λ/Λ', indeed of type $(6,2)$, and write

$$\Lambda=\langle\Lambda',\,e=\frac{a_1e_1+\cdots+a_ne_n}{6},\,f=\frac{b_1e_1+\cdots+b_ne_n}{2}\rangle$$

with $a_i\in\{1,2,3\}$ and $b_1\in\{0,1\}$. We associate a system (m_1,m_2,m_3) with e, a system (m'_1,m'_2,m'_3) with $e'=e+f$ (whose numerator has coefficients a_i+3b_i), and a system (m''_1,m'_2,m''_3) with $2e+f$, which generates the third subgroup of order 6 in Λ/Λ'. Note that m_3,m'_3,m''_3 are non-zero. If $a_i=3$ and $b_i=1$ for some i, then we have $m_1' + m_2' + m_3' \leq 7$, and since denominator 6 is impossible in dimension 7, all $a_i + 3b_i$ must be even (they cannot all be divisible by 3!), which implies $f \equiv 3e \mod \Lambda_0$, hence that Λ/Λ' has order only 6. Hence $b_i = 0$ whenever $a_1 = 1$ or 2. But $2e$ is congruent modulo Λ' to a vector of the form $\frac{\pm e_1 \pm \cdots \pm e_{m_1+m_2}}{3}$, so that $e_1, \dots, e_{m_1+m_2}, 2e + f$ define a lattice of index 6 in dimension $8 - m_3 < 8$, a contradiction.

[The superfluous cases where Λ/Λ' would be of type $(4,4)$ or $(4,2,2)$ can be dealt with using the remark they define several structures of type $(4,2)$ and that in each case, a vector with denominator 4 has exactly 7 components not divisible by 4.]