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ABSTRACT. We compare for an n-dimensional Euclidean lattice Λ the smallest possible values of the product of the norms of n vectors which either constitute a basis for Λ (Hermite-type inequalities) or are merely assumed to be independent (Minkowski-type inequalities). We improve on 1953 results of van der Waerden in dimensions 6 to 8 and prove partial result in dimension 9.

1. INTRODUCTION

We consider a Euclidean space E of dimension n and *lattices* $\Lambda, \Lambda', \dots$, that is discrete subgroups of rank n of E. For $x \in E$, we define the *norm of* x by $N(x) = x \cdot x$ (the square of the traditional $\|x\|$). The *determinant* $\det(\Lambda)$ *of* Λ is the determinant of the *Gram matrix* $\mathrm{Gram}(e_i \cdot e_j)$ of any basis $\mathcal{B} = (e_1, \dots, e_n)$ for Λ. We also define the *minimum of* Λ as $\min \Lambda = \min_{x \in \Lambda \smallsetminus \{0\}} N(x)$, and its *Hermite invariant* $\gamma(\Lambda) = \frac{\min \Lambda}{\det(\Lambda)^{1/n}}$. The *Hermite constant for dimension* n is $\gamma_n = \sup_\Lambda \gamma(\Lambda)$. (Theorem 1.1 below shows that γ_n exists.)

For a lattice Λ in E, define $H_b(\Lambda)$ and $M(\Lambda)$ as

$$\frac{\min N(e_1) \cdots N(e_n)}{\det(\Lambda)}$$

on bases $(e_1, \dots, e_n)$ for Λ, and independent vectors of Λ, respectively. Set

$$Q_b(\Lambda) = \frac{H_b(\Lambda)}{M(\Lambda)}\,.$$

Hermite, in a series of letters to Jacobi, then Minkowski, in his book *Geometrie der Zahlen*, obtained the following bounds:

Theorem 1.1. *For any n-dimensional lattice Λ, we have*

$$H_b(\Lambda) \le \left(\frac{4}{3}\right)^{n(n-1)/2} \quad \text{and} \quad M(\Lambda) \le \gamma_n^n\,.$$

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[Note that (using an argument of density) Minkowski proved a linear bound for γ_n whereas Hermite's (derived from the bound for $H_b(\Lambda)$) is exponential.]

Proofs of the theorem above can be read in [M], Theorems 2.2.1 and 2.6.8. The proof of Minkowski's theorem given there makes use of a deformation trick, useful in our context: one proves that the local maxima of M are attained on *well-rounded lattices*, that is lattices L having n independent minimal vectors, so that $M(L) = \gamma(L)^n$ is bounded from above by γ_n^n. We may of course chose e_1 minimal among non-zero vectors, then e_2 minimal among vectors not proportional to e_1, etc., whence the name of *theorem of successive minima* generally given to Minkowski's theorem.

It is well known (and we shall recover this fact below) that for a lattice Λ of dimension $n \leq 4$, successive minima constitute a basis for Λ except possibly if Λ is the 4-dimensional centred cubic lattice, for which index 2 may occur. Since this lattice possesses a basis of minimal vectors, we have $M(\Lambda) = H_b(\Lambda)$ up to dimension 4. This is no longer true for $n > 4$, as shown by centred cubic lattices.

In his 1953 *Acta Mathematica* paper [vdW], van der Waerden gives a recursive formula for a bound for $\frac{H_b}{M}$ in dimensions $n \geq 4$. In a visit to Bordeaux (October, 2008), Achill Schürmann pointed out to me that van der Waerden's formula may be given the "closed" form below:

Theorem 1.2. *For $n \geq 4$, we have $Q_b(\Lambda) \leq \left(\frac{5}{4}\right)^{n-4}$.*

He also put forward the conjecture (based on properties of the Voronoi cones) that the bound $\frac{n}{4}$ could hold for $4 \leq n \leq 8$, a better bound than van der Waerden's for $n = 6, 7, 8$. This is the main theorem we are going to prove.

Theorem 1.3. *For $4 \leq n \leq 8$, we have $Q_b(\Lambda) \leq \frac{n}{4}$, and equality is needed if and only if Λ is a centred cubic lattice.*

The choice of an orthonormal basis identifies E with $\mathbb{R}^n$ equipped with his canonical basis $\mathcal{B} = (e_1, \dots, e_n)$, which generates the lattice $\mathbb{Z}^n$. Centred cubic lattices are the lattices which are similar to

$$C_n := \langle \mathcal{B}, e \rangle \quad \text{where} \quad e = \tfrac{e_1 + \cdots + e_n}{2}\,,$$

which can be viewed as a lift of the (unique) $[n, 1, n]$-binary code. We can define similarly the canonical lift Λ_C of any binary code C of weight $\mathrm{wt}(C) \geq 4$, obtaining this way a lattice Λ_C of minimum 1. Taking for C the unique $[9, 2, 6]$-(binary) code C_9, with generator matrix

$$\left(\begin{smallmatrix} 1&1&1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1&1&1 \end{smallmatrix}\right)$$

and weight distribution (6^3), we again obtain a lattice with

$$Q_b(\Lambda) = \left(\tfrac{6}{4}\right)^2 = \tfrac{9}{4}\,.$$

This shows that the statement of Theorem 1.3 does not extend as it stands in dimensions $n \geq 9$.

The proofs of Theorem 1.3 for certain codes that I give below are often valid beyond dimension 8. This suggests that the bound $\frac{n}{4}$ is still valid for $n = 9$. This is the conjecture below, which I partially prove in the next theorem. However in order that this paper should not have an unreasonable length I did not try to prove all cases; see Proposition 1.5 and Remark 6.2.

Conjecture 1.4. *For $n = 9$, we have $Q_b(\Lambda) \leq \frac{9}{4}$, and equality is needed if and only if Λ is either a centred cubic lattice or is similar to the canonical lift of C_9.*

Theorem 1.5. *Conjecture 1.4 is true if Λ contains a sublattice Λ' generated by a frame of successive minima for Λ which satisfies one of the following conditions:*

(1) *Λ/Λ' is 2- or 3-elementary.*
(2) *$[\Lambda : \Lambda'] = 4$.*
(3) *$[\Lambda : \Lambda'] \geq 9$.*

Enlarging the code C_9 with a column $\binom{0}{1}$, we obtain the unique *odd* $[10, 2, 6]$-binary code C_{10}; this has weight distribution $(6 \cdot 7^2)$, and its lift has $Q_b = \frac{6}{4} \cdot \frac{7}{4} = \frac{21}{8} > \frac{10}{4}$; and lifting convenient binary codes of length n and dimension 2 indeed suffices to show that the bound $\frac{n}{4}$ no longer holds beyond $n = 9$.

Here is an outline of the method used to prove Theorems 1.3 and 1.5. For every lattice $\Lambda \subset E$, we denote by Λ' a lattice having as a basis a frame $(e_1, \dots, e_n)$ of successive minima for Λ and by d the annihilator of Λ/Λ'. We define the *maximal index $\imath(\Lambda)$ of* Λ as the maximal value of the index $[\Lambda : \Lambda']$ for Λ' as above.

Given d we may write

$$\Lambda = \langle \Lambda', f_1, \dots, f_k \rangle$$

for vectors f_i of the form $f = \frac{a_1 e_1 + \dots a_n e_n}{d}$. The collection of the n-tuples $(a_1, \dots, a_n)$ modulo d defines a $\mathbb{Z}/d\mathbb{Z}$-code canonically associated with (Λ, Λ'). These codes are classified for $n \leq 8$ in [M1], where I extended previous work by Watson, Ryshkov and Zahareva; [Wa], [Ry], [Za]), and for $n = 9$ in [K-M-S]. The proof of Theorems 1.3 and 1.5 heavily relies on the classification of these $\mathbb{Z}/d\mathbb{Z}$-codes (though some general inequalities will sometimes allow us to skip a detailed case-by-case analysis): we shall calculate for each admissible code C an upper bound of $\frac{H_b(\Lambda)}{M(\Lambda)}$ for $\Lambda \in C$ and check that $\frac{n}{4}$ is attained only on codes defining the lattices listed in these theorems.

The bounds we shall prove for a given code are scarcely optimal, and a closer look will show that they are not optimal whenever they are not sharp on well-rounded lattices. Probably the exact bounds on all codes are attained only on well-rounded lattices. An *a priori* proof of this result would considerably simplify our proofs.

It should be noted that the results of [M1] were obtained essentially by hand: we made use of a computer only to prove the existence of some particular codes, which does not matter for this paper. So Theorem 1.3 will be proved within the frame of "classical" mathematics.

This is no longer true for dimension 9. Though classification details of $\mathbb{Z}/d\mathbb{Z}$-codes can (or could) be skipped for small values of d, I do not see any way of avoiding the heavy calculations using linear programming packages performed in [K-M-S] to prove that only index 16 need be considered if $[\Lambda : \Lambda'] > 12$. This problem of large indices shows up from dimension 7 onwards. In [Wa], Watson proved that if $n = 7$ and $[\Lambda : \Lambda'] > 5$, then $\Lambda \sim \mathbb{E}_7$ (and Λ/Λ' is 2-elementary), and stated an analogue for dimension 8, for which a proof can be read on my home page: if $n = 8$ and $[\Lambda : \Lambda'] > 8$, then $\Lambda \sim \mathbb{E}_8$ (and Λ/Λ' is elementary of order 3^2 or 2^4).

After having recalled in Section 2 some general facts on Watson's index theory, we establish in Section 3 sharp bounds for $Q_b(\Lambda)$ when Λ/Λ' is 2-elementary. Then Section 4 is devoted to dimensions $n \leq 7$, and index 3, and to some cases of index 4. Dimension 8 is dealt with in Section 5 after having proved complements on index 4. This will complete the proof of Theorem 1.3. Theorem 1.5 is then proved in Section 6.

Actually the reference to a basis in the definition of H_b is not pertinent: in [M-S] is displayed an example of a 10-dimensional lattice L which is generated by its minimal vectors but has no basis of minimal vectors, so that $Q_b(L)$ is strictly larger than one, though it would be reasonable to consider that successive minima suffice to describe the behaviour of L.

We may define as follows an invariant H_g for a lattice Λ.

For every finite set $\mathcal{G}$ of generators of Λ, take the maximum $M_{\mathcal{G}}(\Lambda)$ of the products $N(e_1)\cdots N(e_n)$ on all systems of independent vectors $e_1,\dots,e_n$ extracted from $\mathcal{G}$, and define $H_g(\Lambda)$ as the lower bound (indeed, a minimum) of $M_{\mathcal{G}}(\Lambda)$ on all generating sets $\mathcal{G}$.

Finally let $Q_g(\Lambda) = \frac{H_g(\Lambda)}{M(\Lambda)}$.

We clearly have $H_g(L) = M(L)$ for the lattice L above. It will turn out that for dimensions $n \le 8$ the exact bounds for Q_b of Theorem 1.3 is also the exact bounds for Q_g (and also for $n = 9$ under Conjecture 1.4).

2. Some background

The basic methods and results on Watson's *index theory* can be read in [M1] and [K-M-S]. Here we recall a few facts that will be used all along this paper, beginning with *Watson's identity*, the most fruitful tool for what follows, the (simple) proof of which is left to the reader.

2.1. Watson's identity.

Proposition 2.1. (Watson) *Let $\mathcal{B} = (e_1, \dots, e_n)$ be a basis for E and let $a_1, \dots, a_n$ and $d > 1$ be integers. For $\lambda \in \mathbb{R}$, let $\operatorname{sgn}(\lambda) = -1$, 0 or 1 according as λ is negative, positive or zero. Let $e = \frac{a_1 e_1 + \cdots + a_n e_n}{d}$ .Then*

$$\sum |a_i|\big(N(e - \operatorname{sgn}(a_i)\, e_i) - N(e_i)\big) = \left(\Big(\sum_{i=1}^{n} |a_i|\Big) - 2d\right) N(e)\,. \quad \square$$

Definition 2.2. In the sequel we denote by $\mathcal{B} = (e_1, \dots, e_n)$ a basis for E and by Λ' the lattice it generates. With the data above, we set $A = \sum_j |a_j|$. For $i \ge 0$ we denote by S_i the set of subscripts j (or of vectors e_j) for which $|a_j| = i$ and set $m_i = |S_i|$, and define $m \le n$ by $m = \sum_{i \ne 0} m_i$. We also set $T = \dfrac{e_1 + \cdots + e_n}{d}$.

We say that *Watson's condition holds* if $A = 2d$ and the a_i are non-zero (i.e., if $A = 2d$ and $m = n$).

Proposition 2.3. *Assume that Watson's condition holds. Then:*

(1) *We have $N(e - \operatorname{sgn}(a_i)e_i) = N(e_i)$ for all i.*
(2) *We have $|a_i| \le \frac{d}{2}$ for all i.*
(3) *If $(e_1, \dots, e_n)$ is a frame of successive minima for*
$$\Lambda := \langle \Lambda', e \rangle \;=\; \cup_{k \bmod d}\; ke + \Lambda'\,,$$
the e_i have equal norms.
(4) *If moreover $m_1 \ge 1$, then $H_b(\Lambda) = M(\Lambda)$.*

Proof. Negating some e_i if need be, we may assume that all a_i are positive.

(1) Since the right hand side in Watson's identity is zero, we have $a_i(N(e - e_i) - N(e_i)) = 0$ for all i.

(2) If a_i is larger than $\frac{d}{2}$ for some i, then replacing e by $e - e_i$ in Watson's identity changes A into $A + (d - 2a_i) < 2d$.

(3) Suppose that $N(e_i) < N(e_{i+1})$ for some i. By (2), replacing e_{i+1} by $e - e_i$, we still have a system of independent vectors, with

$N(e-e_i)=N(e_i)$ by (1), which contradicts the fact that $(e_1,\dots,e_n)$ is a frame of successive minima.

(4) Choose i with $a_i=1$. Then replacing e_i by $e-e_j$ for some $j\ne i$, we obtain a basis for Λ made of vectors of norm $\min\Lambda'$.
[Note that the equality $H_g(\Lambda)=M(\Lambda)$ holds even if $m_1=0$.] $\square$

2.2. **A crude bound.** We consider a frame $\mathcal{B}=(e_1,\dots,e_n)$ of successive minima for a lattice Λ, denoting by Λ' the lattice with basis $\mathcal{B}$, and assume that Λ/Λ' is cyclic of order d, writing $\Lambda=\langle\Lambda',e\rangle$ with $e=\frac{a_1e_1+\dots a_ne_n}{d}$. Reducing modulo d the numerator of e and negating some e_i, we may and shall assume that the a_i satisfy $0\le a_i\le\frac{d}{2}$.

Proposition 2.4. *With the hypotheses above, we have*

$$N(e)\le\frac{\sum_{i=1}^n a_iN(e_i)\sum_{j=i}^n a_j}{d^2}\,,$$

and in particular,

$$N(e)\le\frac{\sum_i m_i(m_i+1)/2\cdot i^2+\sum_{i<j} m_im_j\cdot ij}{d^2}\,N(e_n)\,.$$

Proof. Just develop the expression of e, and observe that if $i<j$ (because the e_i are successive minima), we have $N(e_j-e_i)\ge N(e_j)$, hence

$$2\,e_i\cdot e_j=N(e_i)+N(e_j)-N(e_j-e_i)\le N(e_i)$$

$\square$

We shall use this crude bound to bound the norm of vectors $e-e_i$ or $e-e_i-e_j$ by successive applications of Watson's identity, and also sometimes prove improvements for a convenient choice of e, as in Lemma 3.1 below. We quote as a corollary the case of equal a_i, the proof of which is and easy consequence of the inequalities

$$(n-k+1)N(e_k)+kN(e_{n-k+1})\le\tfrac{n+1}{2}\left(N(e_k)+N(e_{n-k+1})\right).$$

for $k=1,\dots,\lfloor\frac{n}{2}\rfloor$:

Corollary 2.5. *If* $e=\dfrac{e_1+\dots+e_n}{d}$, *then*

$$N(e)\le\frac{n+1}{2d^2}\sum_{i=1}^n N(e_i)\le\frac{n(n+1)}{2d^2}\,N(e_n)\,.\quad\square$$

When constructing bases for Λ from a frame $\mathcal{B}$ of successive minima, we shall replace some vectors e_i of $\mathcal{B}$, including e_n, by convenient vectors $f_i\in\Lambda\smallsetminus\Lambda'$. We shall then have to bound a product $\prod\frac{N(f_i)}{N(e_i)}$, where in practice, i is the largest subscript in the support of the numerator of f_i.

Our results solely depend on the similarity class of Λ. For these reason we shall often assume from Section 4 onwards that Λ is scaled so that $N(e_n) = 1$.

3. 2-elementary quotients

In this section we apply the theory of binary codes to obtain bounds for $H_b(\Lambda)/M(\Lambda)$ when Λ/Λ' is 2-elementary. The results we obtain together with those of Section 2 suffice to prove Theorem 1.1 in dimensions $n \leq 6$.

3.1. **Index 2.** In this subsection we assume that $\Lambda = \langle \Lambda', e \rangle$ where $e = \dfrac{e_1 + \cdots + e_n}{2}$.

Lemma 3.1. *Let* $\mathcal{S} = \left\{ \dfrac{e_1 \pm e_2 \cdots \pm e_n}{2} \right\}$.

(1) *There exists $x \in \mathcal{S}$ of norm $N(x) \leq \dfrac{N(e_1) + \cdots + N(e_n)}{4}$, and equality is needed if and only if the e_i are pairwise orthogonal.*
(2) *If all vectors in $\mathcal{S}$ have a norm $N \geq \max N(e_i)$, then we have $N(x) \leq \frac{n}{4} \max N(e_i)$, and equality holds if and only if the e_i have equal norms and Λ is the centred cubic lattice constructed on the e_i.*

Proof. Negating some e_i if need be, we may assume that e is the shortest of the vectors $\frac{e_1 \pm e_2 \cdots \pm e_n}{2}$. We thus have $N(e - e_i) \geq N(e)$ for $i = 1, \ldots, n$. Summing on i and applying Watson's identity for e, we obtain

$$nN(e) \leq \textstyle\sum_i \big(N(e - e_i) - N(e_i)\big) + \sum_i N(e_i) = (n-4)N(e) + \sum_i N(e_i)\,,$$

i.e., $N(e) \leq \frac{N(e_1) + \cdots + N(e_n)}{4}$.

If equality holds, we must have $N(e - e_i) = N(e)$ for all i. Watson's identity for $e - e_i$, which reads

$$\big(N(e) - N(e_i)\big) + \textstyle\sum_{j \neq i} \big(N(e - e_i - e_j) - N(e_j)\big) = (n-4)N(e - e_i)\,, \quad (*)$$

implies $\sum_{j \neq i} N(e - e_i - e_j) = (n-1)N(e)$. Since $N(e - e_i - e_j) \geq N(e)$, the $(n-1)$ terms $N(e - e_i - e_j)$ must be equal to $N(e)$ for all distinct subscripts i, j. The identity

$$N(e - e_i - e_j) + N(e) = N(e - e_i) + N(e - e_j) + 2e_i \cdot e_j$$

then shows that all scalar products $e_i \cdot e_j$ must be zero.

The converse is clear. This completes the proof of (1).

Still assuming that e has the smallest norm on $\mathcal{S}$, we may assume that we have $N(e_1) \le \cdots \le N(e_n)$. We then clearly have

$$\frac{N(e_1)+\cdots+N(e_n)}{4} \le \frac{n}{4}\,N(e_n)\,,$$

and the inequality is strict unless all e_i have the same norm as e_n. Then Λ is a centred cubic lattice, and conversely centred cubic lattices satisfy $N(x) = \frac{n}{4}\,N(e_i)$ for all $x \in \mathcal{S}$. $\square$

Corollary 3.2. *If a lattice Λ contains to index $\imath = 2$ a sublattice Λ' generated by successive minima of Λ, then $\dfrac{H_b(\Lambda)}{M(\Lambda)}$ is bounded from above by $\frac{n}{4}$, and equality holds if and only if Λ is a centred cubic lattice.*

Proof. Just apply Lemma 3.1 to a frame of successive minima $e_1, \dots, e_n$ for Λ generating a lattice of index 2 in Λ: $(e_1, \dots, e_{n-1}, e)$ is then a basis for Λ, so that $\dfrac{H_b(\Lambda)}{M(\Lambda)} = \dfrac{N(e)}{N(e_n)}$. $\square$

Remark 3.3. Formula $(*)$ above shows that when $n = 4$, all vectors e_i, e, $e - e_i$, $e - e_i - e_j$ have the same norm. The remaining of the proof of Lemma 3.1 then shows that Λ must be a centred cubic lattice.

Remark 3.4. (Watson) Let Λ/Λ' be cyclic of order 4, with $\Lambda = \langle \Lambda', e\rangle$,

$$e = \tfrac{e_1+\cdots+e_{m_1}+2(e_{m_1+1}+\cdots+e_{m_1+m_2})}{4} = \tfrac{e'+e_{m_1+1}+\cdots+e_{m_1+m_2}}{2}\,,$$

$e' = \frac{e_1+\cdots+e_{m_1}}{2}$. Then Watson's identity shows that $m_1 > 4$ implies $n \ge 7$, and Remark 3.3 shows that if $m_1 = 4$, then we must have $m_2 \ge 3$, hence again $n \ge 7$, and that if $m_1 = 4$ and $n = 7$, then e is minimal. This last conclusion holds more generally under Watson's condition if some coefficient a_i is equal to $\frac{d}{2}$, since we may then apply Watson's identity to $e' = e - e_i$ instead of e.

3.2. **Binary codes and 2-elementary quotients.** In this subsection we consider a pair of lattices Λ and $\Lambda' \subset \Lambda$ such that Λ/Λ' is 2-elementary of order 2^k and $\min \Lambda = \min \Lambda'$. We choose a basis $\mathcal{B} = (e_1, \dots, e_n)$ for Λ' and denote by C the binary code (of length n and dimension k) defined by $(\Lambda, \Lambda', \mathcal{B})$. Since $\min \Lambda = \min \Lambda'$, C has weight $w \ge 4$.

Proposition 3.5. *Assume that $\mathcal{B}$ is a frame of successive minima for Λ.*

(1) *We have $\dfrac{H_g(\Lambda)}{M(\Lambda)} \le \dfrac{\min\{\mathrm{wt}(\alpha_1)\cdots\mathrm{wt}(\alpha_k)\}}{4^k}$, where the minimum is taken over all bases $\alpha_1, \dots, \alpha_k$ for C.*

(2) *Assume that C is irreducible (which implies that the support of C is the whole set $\{1, \dots, n\}$). Then if equality holds in (1), the e_i have equal norms.*

(3) *If $k \le 2$ the conclusions of (1) and (2) hold for* $\dfrac{H_b(\Lambda)}{M(\Lambda)}$.

Proof. (1) By Corollary 3.2 we can lift each word α_i to a vector $x_i \in \Lambda$ of norm $N(x_i) \le \frac{\mathrm{wt}(\alpha_i)}{4}$. A set S of n independent vectors extracted from the set $\{x_i, e_i\}$ consists of $\ell \le k$ vectors x_i and $n-\ell$ vectors e_j, satisfying the condition: for every i there exists $j = j[i]$ in the support of α_i such that e_j is not in S. It is then clear that we have

$$\frac{\prod_{x \in S} N(x)}{\prod_{1 \le i \le n} N(e_i)} \le \prod_{i=1}^{\ell} \frac{N(x_i)}{N(e_{j[i]})} \le \prod_{i=1}^{\ell} \frac{wt(\alpha_i)}{4} \le \prod_{i=1}^{k} \frac{wt(\alpha_i)}{4}\,.$$

(The last inequality results from the lower bounds $\mathrm{wt}(\alpha_i) \ge 4$, which hold because $\min \Lambda = \min \Lambda'$.)

(2) Since C is irreducible, we may order $\alpha_1, \dots, a_k$ so that the supports of α_i and α_{i+1} have a non-empty intersection for every $i < k$. By Lemma 3.1, we have $N(e_i) = N(e_j)$ whenever i, j both belong to the support of some α_ℓ, and the hypothesis $\mathrm{Supp}(\alpha_i) \cap \mathrm{Supp}(\alpha_{i+1}) \ne \emptyset$ proves (2).

(3) If $k = 1$, we obtain a basis for Λ by replacing any e_i with $i \in \mathrm{Supp}(x_1)$ by x_1. This method clearly extends (by induction) to all codes satisfying the condition

$$\forall\, i,\ \mathrm{Supp}(\alpha_i) \not\subset \cup_{j \ne i} \mathrm{Supp}(\alpha_j)\,.$$

This remark applies in particular to codes of dimension 2, for if there were an inclusion, say, $\mathrm{Supp}(\alpha_2) \subset \mathrm{Supp}(\alpha_1)$, we could replace α_1 by $\alpha_1 + \alpha_2$, a word of smaller weight. ☐

Remark 3.6. Two vectors e_i and e_j are necessarily orthogonal if i, j belong to the support of some α_ℓ (or of some word of weight 4), but this is not general. For instance, if C is the code $[8, 2, 5]$-code

$$C_8 = \left(\begin{smallmatrix} 1&1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1&1 \end{smallmatrix}\right)$$

(see Subsection 3.3 below), the e_i must have equal norm and be pairwise orthogonal if $i < j \le 5$ or $4 \le i < j$, but we still have $\frac{H_b}{M} = \frac{25}{16}$ on the lifts of C_8 provided that $|e_i \cdot e_j|$ be small enough for $i = 1, 2, 3$ and $j = 6, 7, 8$ so as to have $N(x) \ge \frac{5}{4}$ for any $x \in \Lambda$ which lifts the weight-6 word $(1^3\, 0^2\, 1^3)$. Thus the lattices Λ which lift C_8 in a given scale (say, $\min \Lambda = 1$) depend on 9 parameters.

3.3. **Dimensions up to 10.** We consider an $[n \le 10, k \ge 1, w \ge 4]$-binary code C. We prove for quotients Λ/Λ' associated with C the bounds for $\frac{H_b(\Lambda)}{M(\Lambda)}$ announced in the introduction ($\frac{n}{4}$ if $4 \le n \le 9$, $\frac{21}{8}$ if $n = 10$), and characterize the cases when equality holds.

We may assume that $k \ge 2$ (since the case when $k = 1$ has been dealt with in Corollary 3.2), that the support of C is the whole set $\{1, \dots, n\}$ (since otherwise we may apply results for dimension $n-1$), and that $w > 4$ (if $wt(\alpha_1) = 4$, we reduce ourselves to the case of $\dim C = k-1$ by considering $\langle \Lambda', x_1 \rangle$ instead of Λ'). Then C contains an even subcode C_0 of dimension $k-1$ and weight $w_0 \ge 6$. (Note however that $|\mathrm{Supp}(C_0)|$ may be strictly smaller than n.)

It is readily verified that for $n \le 8$, every $[n, 2, w \ge 4]$-code contains a word of weight 4, except for the a unique $[8, 2, 5]$-code (the code C_8 of Remark 3.6). This has weight distribution $6 \cdot 5^2$, so that its lifts satisfy $\frac{H_b(\Lambda)}{M(\Lambda)} \le \frac{25}{16} < \frac{n}{4} = 2$. This also proves the existence of a weight-4 word if $k \ge 3$, and completes the proof of Theorem 1.3 for 2-elementary quotients.

Let now $n = 9$ and first $k = 2$. It is again readily verified that codes of weight $w \ge 5$ and support $\{1, \dots, 9\}$ have weight distributions $8 \cdot 5^2$, $6 \cdot 5 \cdot 7$ or 6^4 and that there exists a unique code for each weight distribution, which gives for $\frac{H_b(\Lambda)}{M(\Lambda)}$ the exact bounds $\frac{25}{16}$, $\frac{15}{8}$ and $\frac{9}{4}$, respectively, and proves that if $k = 3$, C must extend the code C_9. It is then easily checked that such an extension by a word of weight 5 (resp. 6) must contain a word of weight 3 (resp. 4).

This completes the proof of Theorem 1.5 for 2-elementary quotients.

Let now $n = 10$ and first $k = 2$. We easily check as above that codes of weight $w \ge 5$ and support $\{1, \dots, 10\}$ have weight distributions $10 \cdot 5^2$, $8 \cdot 5 \cdot 7$, $6 \cdot 5 \cdot 9$, $6 \cdot 7^2$, and $6^2 \cdot 8$. The largest upper bound for $\frac{H_b(\Lambda)}{M(\Lambda)}$ is $\frac{21}{8}$, attained on a unique code, namely

$$C_{10} = \left(\begin{smallmatrix} 1&1&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&1&1&1&1&1 \end{smallmatrix}\right).$$

This also shows that there are exactly two even $[10, 2, w \ge 6]$-codes, namely

$$C_{10a} = \left(\begin{smallmatrix} 1&1&1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&1&1&0 \end{smallmatrix}\right) \quad \text{and} \quad C_{10b} = \left(\begin{smallmatrix} 1&1&1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&1&1&1 \end{smallmatrix}\right)$$

(C_{10a} extends C_9). It is an easy exercise to check that even extensions to $k = 3$ of these codes have weight at most 4, and that each of these codes has a unique odd extension, of weight 5. We obtain this way two $[9, 3, 5]$-codes, with weight distributions $6^3 \cdot 5^3 \cdot 7$ and $6^2 \cdot 8 \cdot 5^4$, so that any lift Λ of one of these codes satisfies the bound

$$\frac{H_b(\Lambda)}{M(\Lambda)} \le \frac{125}{64} < \frac{21}{8}\,.$$

We state below as a proposition our result for dimension 10.

Proposition 3.7. *Let Λ be a 10-dimensional lattice having a frame of successive minima generating a lattice Λ' such that Λ/Λ' is 2-elementary. Then we have*

$$\frac{H_b(\Lambda)}{M(\Lambda)} \le \frac{21}{8}\,,$$

and if equality holds, Λ is a lift of the code C_{10}. □

3.4. **More on index 2.** We return to the notation of the first subsection, Lemma 3.1, (2), but now want for further use to bound the norm of e itself rather than that of some suitably chosen vector in $e+\Lambda$. We write $\Lambda = \langle \Lambda', e\rangle$ with $e = \frac{e_1+\dots+e_n}{2}$, and observe that since $\mathcal{B} = (e_1,\dots,e_n)$ is a frame of successive minima for Λ, all vectors e, $e-e_i$, $e-e_i-e_j$, etc, have a norm larger that $\max_i N(e_i)$.

In this subsection we shall have to consider the Coxeter lattices $\mathbb{A}_5^3$ and $\mathbb{D}_6^+$ and the Coxeter-Barnes lattices $\mathbb{A}_n^2$, $n \ge 7$, for the definitions of which we refer to [M], Sections 4.4, 5.1 and 5.2. Note that $\mathbb{A}_5^3$ and $\mathbb{A}_n^2$ $(n \ge 7)$ are perfect whereas $\mathbb{D}_6^+$ is not.

Notation 3.8. *Set $t = N(e)$, fix a subscript i which minimizes $u := N(e-e_i)$, then a subscript $j \ne i$ which minimizes $v := N(e-e_i-e_j)$, and finally a subscript k which minimizes $w := N(e-e_i-e_j-e_k)$.*

Lemma 3.9. (1) *We have the inequalities*
   (a) $u \le \frac{n+(n-4)t}{n}$;
   (b) $v \le \frac{n+(n-4)u-t}{n-1} \le \frac{2n(n-2)+((n-4)^2-n)t}{n(n-1)}$;
   (c) $w \le \frac{n+(n-4)v-2u}{n-2}$.

(2) *For i as above and any $\ell > 0$, we have*

$$N(e-\ell e_i) \le \ell u - (\ell-1)t + \ell(\ell-1)\,.$$

Proof. The three assertions in (1) result from the Watson identity applied to e, $e-e_i$ and $e-e_i-e_j$, respectively, and (2) from the identity $N(e-\ell e_i) = \ell(e-e_i) - (\ell-1)N(e) + (\ell^2-\ell)N(e_i)$. □

Proposition 3.10. *Assume that $\mathcal{B} = (e_1,\dots,e_n)$ $(n \ge 4)$ is a frame of successive minima for Λ. Then we have*

$$N(e) \le \frac{n(n+1)}{8}\,N(e_n)\,,$$

and for $n = 4$, 5 and 6, we have the better bounds

$$N(e) \le N(e_4)\,,\ N(e) \le \tfrac{5}{2}\,N(e_5)\,,\ \text{and } N(e) \le 4\,N(e_6)\,,$$

respectively. These bounds are optimal and attained uniquely on well-rounded lattices. Moreover, if $n \neq 6$, they are attained on a unique similarity class of lattices.

Proof. Without loss of generality, we may assume that Λ has been rescaled so that $N(e_n) = 1$, which implies that $T := N(e_1) + \cdots + N(e_n)$ is bounded from above by n. We first prove the upper bounds.

The first inequality is merely the crude bound of Proposition 2.4.

For $n \leq 6$, the coefficient of t in the second inequality in (1b) of Lemma 3.9 is negative, so that the inequality $v \geq 1$ implies $t \leq 1$ if $n = 4$ and $t \leq \frac{5}{2}$ if $n = 5$.

The inequalities of Lemma 3.9 do not suffice to prove the proposition if $n = 6$. To deal with this case we use directly the Watson identities relative to e, to the $e - e_i$, and to $e - e_i - e_j$, namely

$$(a) \quad \sum_{i=1}^{6} N(e - e_i) = T + 2N(e)\,,$$

then $\forall\, i$,

$$(b) \quad N(e) + \sum_{j \neq i} N(e - e_i - e_j) = T + 2N(e - e_i)\,,$$

and $\forall\, i, j$,

$$(c)\ N(e - e_i) + N(e - e_j) + \sum_{k \neq i,j} N(e - e_i - e_j - e_k) = T + 2N(e - e_i - e_j)\,.$$

Summing on i in (b) and evaluating $\sum N(e - e_i)$ by (a), we obtain

$$(b') \quad 2N(e) + \sum_{i,j;j \neq i} N(e - e_i - e_j) = 8T\,,$$

and summing on i, j in (c) and dividing out both sides by 2, we get

$$(c') \quad 5 \sum_{\ell} N(e - e_\ell) + \frac{1}{2} \sum_{i,j,k \text{ distinct}} N(e - e_i - e_j - e_k) = 15T + \sum_{j \neq i} N(e - e_i - e_j)\,.$$

Evaluating $\sum_\ell N(e - e_\ell)$, adding (b') and (c') yields

$$12N(e) + \frac{1}{2} \sum_{i,j,k \text{ distinct}} N(e - e_i - e_j - e_k) = 18T \leq 108\,,$$

hence $N(e) \leq \frac{1}{2}\,(108 - 60) = \frac{48}{12} = 4$.

In all cases (including in Proposition 2.4), if equality holds we necessarily have $T = n$, which is equivalent to $\forall\, i$, $N(e_i) = 1$ and shows that Λ must be well-rounded.

If $n = 4$ Λ is a centred cubic lattice by Lemma 3.1.

If $n = 5$, the proof above shows that all vectors e_i, $e - e_i - e_j$, $j \neq i$ have the same norm. A simple calculation will show that these condition determines uniquely the Gram matrix of the e_i once their norm is given, i.e., that Λ is perfect. By the classification of 5-dimensional perfect lattices (see [M], Section 6.4), since Λ is not a root lattice (Λ is not integral when scaled to minimum 2), Λ is similar to $\mathbb{A}_5^3$, and we easily check that $\frac{N(e)}{N(e_i)} = \frac{5}{2}$.

The situation is somewhat similar if $n \geq 7$. The bound for $N(e)$ given in the proposition is attained only if $e_i \cdot e_j = \frac{1}{2}$ for all i and $j \neq i$. Scaling the e_i to norm 2 we recognize the Korkine and Zolotareff Gram matrix for $\mathbb{A}_n$ (with entries $a_{i,i} = 2$ and $a_{i,j} = 1$ off the diagonal). This shows that Λ' is then similar to $\mathbb{A}_n$, and we then have $\min \Lambda = \min \Lambda'$ (by results of Coxeter and Barnes; see [M], Section 5.1). Again Λ is perfect, and we easily check that the value of $\frac{N(e)}{N(e_i)}$ is the convenient one.

Finally if $n = 6$ we content ourselves with an example. Taking $N(e_i) = 3$ and $e_i \cdot e_j = 1$ if $j \neq i$, then we see that Λ' is an integral lattice of minimum 3 for which $N(e) = 12$.
[By a joint theorem with Boris Venkov, the condition $s \geq 16$ characterizes Λ among integral lattices *of minimum* 3 as a scaled copy of $\mathbb{D}_6^+$.] □

4. Dimensions up to seven

In this section we first give a short proof of Theorem 1.3 for dimensions $n \leq 6$, then prove some bounds for lattices of index 4, and finally prove Theorem 1.3 for dimension 7.

4.1. **Dimensions up to 6.**

In this subsection we prove Theorem 1.3 for $n \leq 6$. Recall (Watson; see [M1], Theorem 1.7) that we have $\imath(\Lambda) \leq \gamma_n^{n/2}$.

If $\imath(\Lambda) = 1$, there is nothing to prove. Now one has $\gamma_n^{n/2} \leq 2$ if $n \leq 4$, and the value 2 is attained by $\gamma(\Lambda)$ only if $n = 4$ and Λ is the centred cubic lattice (similar to the root lattice $\mathbb{D}_4$), which has a basis of minimal vectors. This shows that we have $H_b = M$ for all $n \leq 4$.

Next if $\imath(\Lambda) = 2$ (which needs $n \geq 4$), Theorem 1.3 results from Corollary 3.2. This applies to dimension 5 since $\gamma_5^{5/2} = \sqrt{8} < 3$.

For $n = 6$, we have $\gamma_6^3 = 4.618\ldots$, so that we need also consider indices 3 and 4. If $\imath = 3$ we have $Q_b = 1$ by Proposition 2.3, and if $\imath = 4$, we know by Remark 3.4 that Λ/Λ' is 2-elementary. Thus we may apply Proposition 3.5: there is a unique $[6, 2, 4]$-code, it has weight distribution (4^3), so that we again have $Q_b = 1$. (The lifts of this code are similar to the root lattice $\mathbb{D}_6$; see see [M1], Table 11.1).

This completes the proof of Theorem 1.3 in dimensions $n \leq 6$.

4.2. **A bound for index 3.** We consider a lattice Λ, a frame $e_1, \dots, e_n$ of successive minima for Λ and the sublattice Λ' of Λ it generates. We shall prove the strict inequality $Q_b(\Lambda) < \frac{n}{4}$ if $[\Lambda : \Lambda'] = 3$ and $7 \leq n \leq 10$. However we consider for further use the slightly more general situation, of index $d \geq 3$, for which

$$\Lambda = \langle \Lambda', e \rangle \text{ where } e = \frac{e_1 + \dots + e_n}{d}$$

Lemma 4.1. *Recall that $T = \sum_{i=1}^n N(e_i)$. Then we have the identity*

$$\sum_{1 \leq i < j \leq n} N(e - e_i - e_j) = (n-2)T + \frac{n^2 - (4d+1)n + 2d(d+2)}{2} N(e)\,.$$

Proof. Consider Watson's identities relative to e and to the $e - e_i$:

$$\sum_{i=1}^n \big(N(e-e_i) - N(e_i)\big) = (n-2d)N(e)$$

and $\forall\, i$,

$$(d-1)\big(N(e) - N(e_i)\big) + \sum_{j \neq i} \big(N(e-e_i-e_j) - N(e_j)\big) = (n-d-2)N(e-e_i)\,.$$

Summing on i in the second identity and evaluating the right hand side using the first identity, we obtain

$$\begin{aligned} (d-1)nN(e) - (d-1)T + \sum_{i,j;j\neq i} N(e-e_i-e_j) - (n-1)T \\ = (n-d-2)\sum_i N(e-e_i) \\ = (n-d-2)(n-2d)N(e) + (n-d-2)T\,, \end{aligned}$$

from which the required identity follows after dividing out by 2 the coefficients of T and of $N(e)$. $\square$

Lemma 4.2. *There exists among the vectors $e - e_i - e_j$, $1 \leq i < j < n$, a vector x such that*

$$N(x) \leq \frac{2(n^2 - 3n + 1) + \big(n^2 - (4d+1)n + 2d(d+2)\big)\, N(e)}{(n-1)(n-2)}\,.$$

Proof. Since (e_i) is a frame of successive minima for Λ, we have $N(e_i) \leq N(e_n) = 1$ for all i, hence $T \leq n$, and $N(f) \geq N(e_n)$ for all $f \in \Lambda \smallsetminus \Lambda'$. In the identity of Lemma 4.1 the left hand side is a sum of $\frac{n(n-1)}{2}$ terms from which we discard the $(n-1)$ terms $e - e_i - e_n$, obtaining

$$\sum_{1 \leq i < j < n} N(e-e_i-e_j) \leq n(n-2) - (n-1) + \big(n^2 - (4d+1)n + 2d(d+2)\big)\, N(e)\,.$$

Dividing out the right hand side by $\frac{n(n-1)}{2}$ yields the inequality we want to prove for the smallest norm of a vector $e - e_i - e_j$, $i < j < n$. $\square$

Lemma 4.3. *With the hypothesis of Lemma 4.2, assume moreover that we have $n \leq 3d + 1$. Then there exists among the vectors e and $e - e_i - e_j$, $1 \leq i < j < n$, a vector y such that*

$$N(y) \leq \frac{n^2 - 3n + 1}{(2d-1)n - (d^2 + 2d - 1)}\,.$$

In particular if $d = 3$ and $n \leq 10$, or $d = 4$, $m_2 = 0$ and $n \leq 13$, then $\frac{H_b(\Lambda)}{M(\Lambda)}$ is strictly smaller than $\frac{n}{4}$.

Proof. View $N(e)$ as a parameter $t \geq 1$, and denote by $\varphi_{n,d}(t)$ the bound for $N(x)$ proved in Lemma 4.2. The coefficient $\alpha(n, d)$ of t in the numerator of φ, viewed as a function of n, attains its minimum on $\mathbb{R}$ for $n = 2d + \frac{1}{2}$, hence on $\mathbb{Z}$ for $n = 2d$ and $n = 2d + 1$, equal to $-2d^2 + 2d < 0$, and takes for $n = 3d + 1$ the value $-d^2 + 3d < 0$. Thus $\varphi_{n,d}(t)$ is a decreasing function of t on $(1, +\infty)$ and attains its maximum at $t = 1$, which is easily seen to be greater than 1. Since t itself is an increasing function, $\min(t, \varphi_{n,d}(t))$ is bounded from above by the value of t for which $t = \varphi_{n,d}(t)$, say, $\psi(n, d)$, which is the bound given in the Proposition. The comparison with $\frac{n}{4}$ is obvious. $\square$

Proposition 4.4. *With the notation of the lemmas above, assume that we have either $d = 3$ and $7 \leq n \leq 10$, or $d = 4$ and $n \leq 13$. Then $Q_b(\Lambda)$ is strictly smaller than $\frac{n}{4}$.*

Proof. The vector y in Lemma 4.3 is of the form $\frac{a_1e_1 + \cdots + a_{n-1}e_{n-1} + e_n}{d}$, so that $(e_1, \ldots, e_{n-1}, y)$ is a basis for Λ, and the bound of $N(y)$ of Lemma 4.3 is thus a bound for $Q_b(\Lambda)$. $\square$

Remark 4.5. The methods of Proposition 4.4, the proof of which relies on the crude bounds of Proposition 2.4 and Corollary 2.5, can be used more generally to handle the case when $d = 4$ and $m_1 = n - 1$. One can prove this way the bound $Q_b < \frac{9}{4}$ when $n = 9$ and $(m_1, m_2) = (8, 1)$.

4.3. **Some more bounds for index 4.** In this subsection we consider the case when Λ/Λ' is cyclic of order 4. The notation S_1, S_2, m_1, m_2 ($m_1 \geq 4$) is that of Definition 2.2.

Proposition 4.6. *Assume that we have $7 \leq n \leq 10$ and that Λ/Λ' is cyclic of order 4. Then:*

(1) *If $m_1 = 4$, we have $\mathbb{Q}_b(\Lambda) \leq \dfrac{n-3}{4} < \dfrac{n}{4}$.*

(2) *If* $m_1 = 5$, $\mathbb{Q}_b(\Lambda)$ *is bounded from above by* $\frac{9}{8}$ *if* $n = 7$, *and by* $\frac{(2n+5)^2}{320} < 2$ *if* $n = 8, 9, 10$.

Proof. We keep the notation $e_1, \dots, e_n$ for the successive minima, assuming that $N(e_1) \le \dots N(e_n) = 1$, and $\Lambda' = \langle e_i \rangle$. Set $e' = \dfrac{\sum_{i \in S_1} e_i}{2}$, so that $e = \frac{e' + \sum_{j \in S_2} e_j}{2}$, set $\mathcal{S} = \left\{ \frac{\sum_{i \in S_1} e_i}{2} \right\}$, and denote by α (resp. β) the largest subscript $i \in S_1$ (resp. $i \in S_2$). Thus $\alpha = n$ or $\beta = n$. By Lemma 3.1, negating e_i for some $i \in S_2$, we may assume that e has the smallest norm among vectors of $\mathcal{S}$.

Assume first that $n = \alpha$. Then replacing e_n by e, we obtain a basis for Λ for which $Q_b(\Lambda) \le \frac{N(e') + m_2}{4}$. By Proposition 3.10, we have $N(e') \le 1$, hence $Q_b \le \frac{1 + m_2}{4} = \frac{n-3}{4}$ if $m_1 = 4$, and $N(e') \le \frac{5}{2}$, hence $Q_b \le \frac{5/2 + m_2}{4} = \frac{2n-5}{4}$ if $m_1 = 5$.

Assume now that $n = \beta$. We may no longer replace e_n by e since the numerator of e now contains the term $2e_n$. We can instead replace e_α by any vector $e'' \in \mathcal{S}$ to be chosen later and e_n by e, obtaining the upper bound $Q_b(\Lambda) \le N(e'') \cdot N(e)$.

If $m_1 = 4$, we choose $e'' = e'$, and since $N(e') = N(e_\beta)$, we again have $Q_b \le \frac{n-3}{4}$.

If $m_1 = 5$, taking $x = v$ with the notation of Lemma 3.9, (1b), we may achieve $N(e'') \le \frac{15 - 2t}{10}$, hence $N(e'') \cdot N(e) \le \varphi(t) := \frac{(15-2t)(t+m_2)}{40}$. The maximum of φ on $\mathbb{R}$ is attained at $t = t_0 := \frac{15 - 2m_2}{4}$.

If $n = 7$, i.e., $m_2 = 2$, we have $t_0 > \frac{5}{2}$, the bound for t of Proposition 3.10, and since $\varphi(1) < 1$, the maximum of φ on $[1, \frac{5}{2}]$ is $\varphi(\frac{5}{2}) = \frac{9}{8}$.

If $n = 8, 9, 10$, i.e., $m_2 = 3, 4, 5$, we have $t_0 \in (1, \frac{5}{2})$, hence

$$N(e'')N(e) \le \varphi(t_0) = \tfrac{(2n+5)^2}{320} \text{ if } n = 8, 9 \text{ or } 10\,,$$

slightly larger than the bounds we obtained for $n = \alpha$. $\square$

Remark 4.7. The bounds of Proposition 4.6 are optimal if $m_1 = 4$, and if $n = 7$ and $m_1 = 5$, and attained uniquely on well rounded lattices. The bounds for $n = 8, 9, 10$ and $m_1 = 5$ are not optimal, and even the first bound $\frac{2n-5}{4}$, which applies to well-rounded lattices, could be improved, using vectors $e - e_i$ or $e - e_i - e_j$, $i, j \in S_1$.

4.4. **Dimension 7.** We now prove Theorem 1.3 for dimension 7, by inspection of all possible structures of Λ/Λ' when Λ' is generated by a frame of successive minima $e_1, \dots, e_n$ for Λ, that we scale so as to have $N(e_n) = 1$.

We know from [M1] that Λ/Λ' is of one of the types (1), (2), (3), (4), (2^2), (2^3), the latter case occurring only on the similarity class of $\mathbb{E}_7$. Thus there is nothing to prove if $[\Lambda:\Lambda'] = 1\,\text{or}\,8$.

The case of index 2 results from Corollary 3.2, and that of 2-elementary quotients has been dealt with in Subsection 3.3. (There are two primitive codes of weight $w \geq 4$. There weight distributions are $4^2 \cdot 6$ and $4 \cdot 5^2$, so that we have $Q_b = 1, \frac{5}{4}$, respectively.)

The case of index 3 results from Proposition 4.4, which implies $Q_b(\Lambda) \leq \frac{29}{21} = 1.38... < \frac{7}{4} = 1.75\,.$

Consider finally the case when Λ/Λ' is cyclic of order 4. We have $4 \leq m_1 \leq 6$ by Watson's identity 2.1, $Q_b = 1$ if $m_1 = 4$ or 6 by Remark 4.7, and $Q_b \leq \frac{9}{8} < \frac{7}{4}$ if $m_1 = 5$ by Proposition 4.6.

This completes the proof of Theorem 1.3 for all dimensions $n \leq 7$.

The bound above for index 3 is still not optimal, and can be improved by making use also of vectors $e - e_i$ to $\frac{(n-2)(n^2-2n-9}{(n-1)(5n-18)}$ ($\frac{65}{51} = 1.27...$ for $n = 7$, still not optimal).

We summarize in the table below our knowledge on optimal bounds for dimension 7. The lower bounds for cyclic quotients of order 3 and 4 are attained on the Gram matrices $A_{7,3}$ and $A_{7,4}$ displayed after the table below.

If we restrict ourselves to well-rounded lattices, we need not discard the subscript n in the lemmas above. I could then show that $\frac{11}{9}$ $(= 1.22\dots)$ is optimal among well-rounded lattices. This is probably the general exact bound.

1	2	3	4	2^2	2^3
1	$\frac{7}{4}$	$\frac{11}{9} \leq Q_b < \frac{65}{61}$	$\frac{9}{8}$	$\frac{5}{4}$	1

Table 1. Optimal bounds in dimension 7

Here are Gram matrices for lattices which realize $\frac{H_b}{M} = \frac{11}{9}$ and $\frac{H_b}{M} = \frac{9}{8}$ for cyclic quotients of order 3 and 4, respectively:

$$A_{7,3} = \begin{pmatrix} 22 & -6 & 9 & 9 & 9 & 9 & 9 \\ -6 & 18 & 3 & 3 & 3 & 3 & 3 \\ 9 & 3 & 18 & 3 & 3 & 3 & 3 \\ 9 & 3 & 3 & 18 & 3 & 3 & 3 \\ 9 & 3 & 3 & 3 & 18 & 3 & 3 \\ 9 & 3 & 3 & 3 & 3 & 18 & 3 \\ 9 & 3 & 3 & 3 & 3 & 3 & 18 \end{pmatrix}; \qquad A_{7,4} = \begin{pmatrix} 9 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 8 & 2 & 2 & 2 & 0 & 0 \\ 4 & 2 & 8 & 2 & 2 & 0 & 0 \\ 4 & 2 & 2 & 8 & 2 & 0 & 0 \\ 4 & 2 & 2 & 2 & 8 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 8 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 8 \end{pmatrix}.$$

In both cases the minimum is read on the diagonal entries $a_{i,i}$, $i \geq 2$ and $\frac{H_b}{M} = \frac{a_{1,1}}{a_{2,2}}$.

5. Indices 4 and 5 and dimension 8

In this section we complete the proof of Theorem 1.3 by calculating sufficient bounds for $\frac{H_b}{M}$ in dimension 8. However, for further use, we sometimes consider dimensions n which may be greater than 8.

We keep the notation of the previous section: Λ is an n-dimensional lattice and Λ' denotes its sublattice generated by a frame $\mathcal{B} = (e_1, \dots, e_n)$ of successive minima for Λ, and we assume that $N(e_n) = 1$. The notation m_i, S_i, T_i that we use when dealing with cyclic quotients is that of Definition 2.2.

5.1. **Index 4.** Though an *ad hoc*, relatively short proof could be given for $n = 8$, we prove below bounds which also apply to dimension 9.

Proposition 5.1. *Assume that Λ/Λ' is cyclic of order 4. Then if $n \le 9$, and if either $m_1 \le 7$ or $m_1 = n$, then $Q_b(\Lambda) < \frac{n}{4}$.*

Proof. The result has been proved previously if $m_1 = n$ (Proposition 4.4), if $m_1 \le 5$ (Proposition 4.6), and if $n = 7$.

The proof for $m_1 = 6$ is an extension of that of Proposition 4.6 whereas we need sharper inequalities for $m_1 = 7$.

In all cases we assume that the norm of e is minimal among those of the vectors $\frac{e' + \sum_{i \in S_2} \pm e_i}{2}$.

$\underline{m_1 = 6}$ (thus, $m_2 \in \{2, 3\}$). With the notation of Lemma 3.9, we choose $x = v$ if $u \le 2$ and $x = w$ if $u \ge 2$, bounding this way $N(x)N(e)$ by the functions φ_1 and φ_2 below, to to be considered on the interval $[1, 4]$ by Proposition 3.10:

$$u \le 2\ :\ \varphi_1(t) = \frac{(10 - t)(t + m_2)}{20}\ ;\ u \ge 2\ :\ \varphi_2(t) = \frac{(15 - t)(t + m_2)}{40}\,.$$

The maximum on $\mathbb{R}$ of φ_1 is attained at $t_1 = \frac{10 - m_2}{2}$ and that of φ_2 at $t_2 = \frac{15 - m_2}{2}$. Since $m_2 = 2$ or 3, we have $t_1 \in [1, 4]$ and $t_2 > 4$. Hence $\frac{H_b(\Lambda)}{M(\Lambda)}$ is bounded from above by $\varphi_1(t_1)$ if $u \le 2$ and $\max(\varphi_2(1), \varphi_2(4)) = \varphi_2(4)$ if $u \ge 2$, that is, in terms of $n = 6 + m_2$,

$$\varphi_1(t_1) = \tfrac{(n+4)^2}{80} = \tfrac{n}{4} - \tfrac{(10-n)(n-2)+4}{80} \text{ and } \varphi_2(4) = \tfrac{11(n-2)}{40} = \tfrac{n}{4} - \tfrac{22-n}{40}\,,$$

which proves the result (even up to $n = 10$).

To handle lattices with $m_1 \ge 7$ we return to Watson's identity for denominator 4, namely

$$\sum_{i \in S_1} N(e - e_i) + 2 \sum_{i \in S_2} N(e - e_i) = T_1 + 2T_2 + (m_1 + 2m_2 - 8)\, N(e)\,,$$

which implies, since $N(e-e_i) \ge N(e)$ for all $i \in S_2$,

$$\sum_{i\in S_1} N(e-e_i) \le m_1 + 2m_2 + (m_1-8)\,N(e)\,.$$

$\underline{m_1 = 7, m_2 = 1}$. We have $N(e) \le \frac{t+1}{4}$, hence $t \ge 3$, and by the crude estimate, $t \le 7$, hence $u \le 1 + \frac{37}{7} \le 4$. Using $\tilde{e} = \frac{(e'-2e_i)+e_\beta}{4}$ of norm $N(\tilde{e}) \le \frac{2u-t+3}{4}$ instead of e and taking $x = w$, of norm $N(x) \le \frac{21-u-t}{10}$ (Lemma 3.9) we reduce ourselves to bound the function $\varphi(t) = \frac{(21-u-t)(2u-t+3)}{40}$ in the domain defined by the inequalities $1 \le u \le 1 + \frac{3t}{7} \le 4$ and $3 \le t \le 7$. As a function of u, φ' is zero for $u = \frac{39-t}{4} \ge 8 > 4$, so that we have $\varphi(u,t) \le \varphi(1+\frac{3t}{7}, t) = \frac{(35-7)(14-t)}{4\cdot 7^2}$, a decreasing function on $[3,7]$. Its maximum on $[3,7]$ is attained for $t = 3$ and is equal to $\frac{32\cdot 11}{4\cdot 7^2} = \frac{88}{49} < 2$.

$\underline{m_1 = 7, m_2 = 2}$. This time we find $\varphi(t) = \frac{(21-u-t)(2u-t+4)}{40}$, to be considered in the domain $1 \le u \le 1 + \frac{3t}{7}$ and $2 \le t \le 7$. As a function of u, its maximum is $M_1 = \varphi(1+\frac{3t}{7}, t) = \frac{(42-t)(14-t)}{4\cdot 7^2}$. For $t \ge 3$ we have $M_1 \le \frac{429}{196} = 2.18... < \frac{9}{4}$. For $t \in [2,3]$, we use $N(x)N(e) \le \frac{(21-u-t)(t+2)}{40}$, which as a function of t is maximum at $t = 3$ and then equal to $\frac{18-u}{8} < \frac{17}{8} < \frac{9}{4}$. $\square$

This completes the proof of Theorem 1.3 for index 4.

5.2. **Index 5.** We use as for denominator 4 the notation m_1, m_2, S_1, S_2. The cosets modulo Λ' are those of 0, $\pm e$ and $\pm e'$ where

$$e = \frac{\sum_{i\in S_1} e_i + 2\sum_{i\in S_2} e_i}{5} \quad\text{and}\quad e' = \frac{2\sum_{i\in S_1} e_i - \sum_{i\in S_2} e_i}{5}\,.$$

We have $e' \equiv 2e \mod \Lambda'$ so that exchanging e and e' and negating the e_i with $i \in S_2$ if need be, we may assume that we have $m_1 \ge m_2$. Then by Proposition 2.1, we have $(m_1, m_2) = (4,4)$, $(5,3)$ or $(6,2)$ if $n = 8$ (and $(5,4)$, $(6,3)$, $(7,2)$ or $(8,1)$ if $n = 9$), and by Proposition 2.3, Λ has a basis of minimal vectors if $(m_1, m_2) = (6,2)$ (or $(8,1)$).

If $m_1 = m_2 = 4$, by an identity of Zahareva ([M1], Section 9), we have $N(e-e_i) = N(e_i)$ for $i = 5,6,7,8$ and $N(e'-e_i) = N(e_i)$ for $i = 1,2,3,4$. Since (e_i) is a frame of successive minima, we have $N(e_1) \le N(e_8) \le N(e'-e_1) = N(e_1)$, which shows that e_i, $e - e_j, j \ge 4$ and $e' - e_k, k \le 4$ are minimal vectors and that $(e_1, \dots, e_7, e - e_1)$ is a basis of minimal vectors for Λ.

To handle the case when $(m_1, m_2) = (5,3)$, we shall use the crude bound of Proposition 2.4, which reads

$$N(e) \leq \tfrac{m_1(m_1+1)/2+2m_2(m_2+1)}{25}\,,$$

together with Watson's identity

$$\textstyle\sum_{i\in S_1}(e-e_i)+2\sum_{i\in S_2}(e-e_i)=T_1+2T_2+(m_1+2m_2-10)N(e)\,,$$

considering separately the cases $n\in S_1$ and $n\in S_2$.

If $n\in S_1$, using the obvious lower bound $N(e-e_i)\geq N(e_i)$ when $i\in S_2$ or $i=n$, we see that there exists among the vectors $e-e_i$, $i\in S_1\smallsetminus\{n\}$ an x of norm $N(x)\leq 1+\frac{m_1+2m_2-10}{m_1-1}\,N(e)$. Then $(e_1,\dots,e_7,x)$ is a basis for Λ. When $(m_1,m_2)=(5,3)$, the bounds above are

$$N(e)\leq\tfrac{69}{25}\ \text{and}\ N(x)\leq 1+\tfrac{N(e)}{4}\leq\tfrac{169}{100}=1.69<2\,.$$

If $n\in S_2$ (i.e., if $e_n=e_8$) we first observe that we have $e'=\frac{-e+\sum_{i\in S_1}e_i}{2}$, so that some vector $x=e'-e_{i_1}-\dots e_{i_k}$, $i_1,\dots,i_k\in S_1$, has a norm $N(x)\leq\frac{N(e)+m_1}{4}$ (Lemma 3.1). Again $(e_1,\dots,e_7,x)$ is a basis for Λ, and when $(m_1,m_2)=(5,3)$, we have

$$N(x)\leq\tfrac{N(e)+5}{4}\leq\tfrac{69/25+5}{4}=\tfrac{194}{100}<2\,.$$

5.3. **Dimension 8.** We now prove Theorem 1.3 for 8-dimensional lattices, namely that in dimension 8, $Q_b=\frac{H_b}{M}$ is bounded from above by 2, with equality only on centred cubic lattices. We consider the various possible structures of Λ/Λ', and recall from [M1] that if $[\Lambda:\Lambda']>8$ then Λ is similar to $\mathbb{E}_8$, which has a basis of minimal vectors, so that it suffices to consider indices $[\Lambda:\Lambda']\leq 8$, excluding cyclic quotients of order 7 or 8 which do not exist in dimension 8.

The case of 2-elementary quotients has been dealt with in Section 3, so that it suffices to consider quotients Λ/Λ' which are either cyclic of order 3 to 6 or of type $4\cdot 2$ and to show that we then have the strict inequality $Q_b(\Lambda)<2$. We now consider successively the five possible cases for the maximal index of Λ.

- $\imath=3$. This is Proposition 4.4.
- $\imath=4$. This results from Proposition 5.1.
- $\imath=5$. This results from Subsection 5.2.
- $\imath=6$. With the notation S_i,m_i for $i=1,2,3$, we have $\Lambda=\langle\Lambda',e\rangle$ where $e=\frac{\sum_i i\sum_{j\in S_i}e_j}{6}$. Besides e we also consider

$$f=\tfrac{\sum_{i\in S_1}e_i-\sum_{j\in S_2}e_j}{3}\quad\text{and}\quad g=\tfrac{\sum_{i\in S_1}e_i+\sum_{k\in S_3}e_k}{2}\,,$$

in order to apply previous results for denominators 2 and 3. Set $f_i=f-e_i$ if $i\in S_1$ and $f+e_i$ if $i\in S_2$.

There are six $\mathbb{Z}/6\mathbb{Z}$-codes listed in [M1], among which five define lattices having a basis of minimal vectors. (This can be easily checked using Section 9 of [M1], as we did above for $(m_1, m_2) = (4, 4)$ with denominator 5.) The remaining code has $(m_1, m_2, m_3) = (3, 3, 2)$. Since $m_1 + m_3 = 6$, Watson's identity shows that the vectors e_i and f_i for $i \in S_1 \cup S_2$ have equal norm. Consider two subscripts $i, j \in S_2$, and in the basis (e_i) for Λ', replace e_i by f_j. The we obtain a new frame of successive minima, which spans a lattice L such that $\Lambda = \langle L, g \rangle$. We are thus reduced to index 2, and this proves that we have $Q_b(\Lambda) \le \frac{m_1+m_3}{4} = \frac{5}{4}$ in this case.

• $\imath = 8$. Here there are three codes over $\mathbb{Z}/4\mathbb{Z}$, and in each case we have $\Lambda = \langle \Lambda', e, f \rangle$ for vectors e and f of denominators 4 and 2, respectively. In all cases Λ has a basis of minimal vectors: in the first case because e and f are minimal, and in the remaining two cases because these are known lattices, namely a lattice on a Voronoi path $\mathbb{E}_8$—$\mathbb{E}_8$ with $s = 75$ discovered by Watson, and $\mathbb{E}_8$.

This completes the proof of Theorem 1.3. □

6. Beyond dimension 8

In this section we collect various results and remarks concerning dimensions $n > 8$. In the first subsection we prove Theorem 1.5, then extend it to some cases concerning indices between 5 and 8. We then consider in the second subsection some extensions Theorem 1.5 for well-rounded lattices. Finally in a short last subsection we make a few remarks on larger dimensions.

6.1. **Proof of Theorem 1.5.** We now proceed to the proof of Theorem 1.5 by looking successively at the various structures of Λ/Λ' listed in its statement. As usual we restrict ourselves to primitive codes, since otherwise the result follows from the bounds we proved for dimensions $n \le 8$.

Proof. • 2-elementary quotients. These have been dealt with in Subsection 3.3.

• 3-elementary quotients. The case when $[\Lambda : \Lambda'] = 3$ is Proposition 4.4. Otherwise we have $[\Lambda : \Lambda'] = 9$ and there are three admissible ternary codes, listed in Table 6 of [K-M-S], all of which have a basis (w_1, w_2) with $\mathrm{wt}(w_1) = 6$ and $\mathrm{wt}(w_2) = 6, 6$ and 7, respectively. Since $\mathrm{wt}(w_1) = 6$ Watson's identity shows that the lattice generated by Λ' and a lift of w_1 has a basis of successive minima, so that we are reduced to the case of index 3 and dimension $\mathrm{wt}(w_2)$, for which we know a bound for $Q_b(\Lambda)$ which is much smaller than $\frac{9}{4}$.

• Index 4. We need only consider cyclic quotients, classified by pairs (m_1, m_2) with $m_1 \ge 4$ and $m_1 + m_2 = 9$. The bound $Q_b < \frac{9}{4}$ has been proved in Proposition 5.1 if $m_1 = 9$ or $m_1 \le 7$, so that we are left with the case when $(m_1, m_2) = (8, 1)$, for which the methods of Subsection 5.1 do not suffice. This case can be solved by bounding from above the smallest norm of a vector $e - e_i - e_j$, with the same line of proof than that of Lemmas 4.1 to 4.3. The details are left to the reader. Note however that the results of Subsection 5.1 suffice for well-rounded lattices.

• Index > 9. The *PARI-GP* companion file *Gramindex.gp* to [K-M-S] shows that there exists for every lattice a Gram matrix having diagonal entries equal to its minimum, except for the matrix $a9f62$, which acquires such a diagonal after an LLL-reduction. Hence all lattices Λ with $[\Lambda : \Lambda'] \ge 10$, except possibly those having a 2-elementary quotient of order 16, indeed have a basis of minimal vectors, hence satisfy $Q_b(\Lambda) = 1$. (For codes over $\mathbb{F}_2$ one has $Q_b = 1$ or $\frac{5}{4}$.)

• Index 9. We need only consider cyclic quotients. Six codes are displayed in Table 2 of [K-M-S]. For the first four, with $s = 84, 50, 136, 53$, respectively, the file *Gramindex.gp* shows the existence of bases of minimal vectors. For the remaining two, and further in the sequel, departing from our previous convention, we order the e_i choosing successively vectors from S_1, then S_2, ..., and write e as successive sums having denominator 3 namely

$$e = \frac{e' + e_5 + e_6 + e_7 + e_8 + e_9}{3} \text{ with } e' = \frac{e_1 + e_2 + e_3 + 2e_4 + e_8 + e_9}{3}$$

and

$$e = \frac{e' + e_4 + e_5 + e_6 + e_7 + e_8 + e_9}{3} \text{ with } e' = \frac{e_1 + e_2 + 2e_3 - e_4 + e_8 + e_9}{3}.$$

Using Watson's identity we see that in both cases, the successive minima on the support of e' are equal, and that the same property holds for e in the first case (and then $Q_b = 1$) whereas we may apply Proposition 4.4 for dimension 7 in the second case. (There are then 15 minimal vectors, which all lie in Λ' or $\pm e' + \Lambda'$.) □

6.2. **More on index 9.** We now refer to Tables 2 and 6 of [K-M-S], and use the notation $C_{d,i}$, $d = 5$, 6, 7, 8 or $4 \cdot 2$, to denote the i-th class of lattices with Λ/Λ' of type (d) in the corresponding table. Here i runs from 1 to $i(d)$, where $i(5) = 4$, $i(6) = 20$, $i(7) = 8$, $i(8) = 19$, and $i(4 \cdot 2) = 26$.

In this table (as in [M1]) s (resp. s') is the number of pairs of necessary minimal vectors for Λ (resp. Λ'). Thanks to a deformation argument the tables could be constructed using only well-rounded lattices; in our context s (resp. s') is the minimal number of pairs of representatives of the successive minima for Λ (resp. Λ').

Proposition 6.1. *Let Λ be a lattice belonging to a class $C_{d,i}$, $d = 5$, 6, 7, 8 or $4{\cdot}2$. Then if $s > 9$, $Q_b(\Lambda)$ is strictly smaller than $\frac{9}{4}$.* [The number of classes satisfying these conditions are 1, 11, 5, 16, and 23, respectively.]

Proof. We first observe that $s > 9$ implies $s > s'$. This shows that in all cases, the lattice L generated by the successive minima of Λ satisfies $[\Lambda : L] < [\Lambda : \Lambda']$. By inspection (or by [M-S]) we see that L actually has a basis made of successive minima.

As above for index 9 we consider the diagonal entries $a_{i,i}$ in the file *Gramindex.gp*. In all cases we have $a_{1,1} \ge a_{2,2} = \cdots = a_{9,9} = \min \Lambda$. If $a_{1,1} = a_{2,2}$ then Λ has obviously a basis of minimal vectors, which implies $Q_b(\Lambda) = 1$. This applies to denominators 5 and 7.

Otherwise we list its minimal vectors and consider the leading components. If some leading component is equal to 1, we again have a basis of minimal vectors. This holds more generally if the leading components are coprime, which solves one more case with $d = 6$ (and an LLL-reduction then produces explicitly a basis of minimal vectors). If the gcd of the leading components is > 1, then $[\Lambda : L]$ takes one of the values 2, 3 or 4, and we need a closer look at minimal vectors.

• $d = 6$. There remains six classes to consider. For two of them we have $m_1 + m_3 = 4$, so that Λ contains to index 3 a lattice having a basis of successive minima, to which we may apply Proposition 4.4, and in the remaining four cases, we have $m_1 + m_2 = 6$, so that Λ contains to index 2 a lattice having a basis of successive minima, to which we may apply Lemma 3.1, after having checked that we may write $\Lambda = \langle L, \frac{e_{i_1}+\cdots+e_{i_k}}{2}\rangle$ for some $k \le 8$, which then ensures the upper bound $Q_b(\Lambda) \le \frac{8}{2} = 2$.
The worst case is afforded by the code $(2, 4, 3)$, for which we write

$$e = \frac{(e_1 + e_2 + 2e_3 - e_4 - e_5 - e_6)/3 + e_4 + e_5 + e_6 + e_7 + e_8 + e_9}{2}\,,$$

obtaining the (indeed strict) bound $Q_b(\Lambda) \le \frac{7}{4}$.

• $d = 8$. There remains two classes to consider, with corresponding codes of type $(2, 4, 2, 1)$ (matrix $a9j8$) and $(3, 1, 3, 2)$ (matrix $a9s8$).

In the first case we set $e' = \frac{e_1+e_2+e_7+e_8}{2}$ and $L = \langle \Lambda', e'\rangle$ and write $e = \frac{e'+e_3+\cdots+e_8+2e_9}{4}$, so that we are reduced to the case of a cyclic quotient of order 4 in dimension 8.

In the second case we set $e' = \frac{e_1+e_2+e_3-e_5-e_6-e_7+2e_4}{4}$ and write $e = \frac{e'+e_5+e_6+e_7+e_8+e_9}{2}$ so that we are reduced to index 2 in dimension 6.

• $d = 4 \cdot 2$. There remains five classes to consider, for which Table 7 of [K-M-S] displays a representation $\Lambda = \langle \Lambda', e, f\rangle$ with $4e$ and $2f$ in Λ'. In all cases (matrices $a9g42$, $a9j42$, $a9r42$, $a9s42$ and $a9t42$) the support of e is of length 7 or 8 and that of f of length 4, so that writing $\Lambda = \langle L, e\rangle$, we are reduced to the case of index 4 in dimension 7 or 8.

This completes the proof of the proposition. $\square$

Remark 6.2. The bound $Q < \frac{9}{4}$ also holds for all quotients Λ/Λ' of type $4 \cdot 2$. Indeed in the three cases where $s = 9$ in Table 7 of [K-M-S], f has a support of length 5, which implies $Q_b(\Lambda) \le \frac{5}{4}\,B$ where B is the bound previously obtained for cyclic quotients of order 4 with $(m_1, m_2) = (5, 2)$, $(6, 2)$ and $(7, 1)$, namely $\frac{9}{8}$, $\frac{66}{40}$ and $\frac{88}{49}$, that is $Q \le \frac{45}{32} < 1.41$, $Q \le \frac{33}{16} < 2.1$ and $Q \le \frac{110}{49} = 2.24...$, respectively. The exact bounds are probably much smaller.

Taking into account Proposition 6.1 and Remark 6.2, we are left with $3 + 9 + 3 + 3 = 18$ unsolved cases, corresponding to cyclic quotients Λ/Λ' of orders $5, 6, 7, 8$, that we list below:
$d = 5$: $m_1 = 5, 6, 7$.
$d = 6$: $m_3 = 0$, $m_1 = 5$; $m_3 = 1$, $m_1 = 4, 5, 6$; $m_3 = 2$, $m_1 = 3$, 4, 5, 6, 7.
$d = 7$: $(m_1, m_2, m_3) = (4, 3, 2)$, $(5, 2, 2)$, $(4, 2, 3)$.
$d = 8$: $(m_1, m_2, m_3, m_4) = (3, 4, 2, 0)$, $(3, 3, 2, 1)$, $(3, 2, 2, 2)$.
To deal with these eighteen remaining cases would make this paper unreasonably long. We consequently end here general proofs, though some more cases could have been solved along the line of Remark 6.2.

6.3. **Well-rounded, 9-dimensional lattices.** In this subsection we consider well-rounded lattices, with $[\Lambda : \Lambda'] = 5$ or 7.

Proposition 6.3. *Let Λ be a well-rounded lattice of dimension 9 and maximal index 5 or 7. Then $Q_b(\Lambda)$ is strictly smaller than $\frac{9}{4}$.*

Proof. We shall write down a detailed proof for $d = 5$, and leave to the reader the case of index 7, for which it suffices to mimic the previous case. The method consists in applying Watson's identity and using the crude estimate 2.4 to bound $N(e)$.

In all cases the ordering of the vectors e_i does not matter, since they all have the same norm, that we fix equal to 1. We order them as we did above for index 9,

Thus let $[\Lambda : \Lambda'] = 5$, write as usual $n = m_1 + m_2$, and assume that $m_1 \ge m_2$. Fix a subscript i in $\{1, \dots, n\}$, then a subscript $j \ne i$ in S_1. The vectors $e_k, k \ne j$ and $e - e_i$ then constitute a basis for Λ, so that we have $Q_b(\Lambda) \le N(e - e_i)$.

Watson's identity, which reads
$\sum_{k\in S_1} N(e_k) + 2\sum_{k\in S_2} N(e_k) = (m_1 + 2m_2) + (m_1 + 2m_2 - 10)N(e)\,,$
shows that there exists i such that

$$N(e-e_i) \leq 1 + \tfrac{m_1+2m_2-10}{m_1+2m_2}\,\mathrm{N(e)}\,,$$

whereas Proposition 2.4 gives the bound

$$N(e) \leq \tfrac{2n(n+1)-m_1(4n+3-m_1)/2}{25}\,.$$

For given n, both $N(e)$ and its coefficient are decreasing functions of m_1, so that $N(e-e_i)$ is bounded from above by its value at $\lfloor \frac{n+1}{2} \rfloor$.

For $n=9$ we obtain the bound $Q_b(\Lambda) \leq 1 + \frac{3}{13}\frac{95}{25} = \frac{122}{65} < \frac{9}{4}$.

The same argument applies to dimension 7, and the large denominator (49 instead of 25) in $N(e)$ yields in all cases an upper bound far below $\frac{9}{4}$. □

Most of the proofs we gave all along this paper for dimensions $7, 8, 9$ could have been made simpler if we had restricted ourselves to well-rounded lattices, and we can even very often easily check that the bounds we obtained are not optimal every time we had to take into account the place of e_n with respect to the subsets $S+i$ relative to various cyclic components. This supports the following conjecture:

Conjecture 6.4. *In all dimensions, the maxima of Q_b and Q_g are attained on well-rounded lattices*

6.4. **Beyond dimension 9.** We just want to consider 2-elementary quotients Λ/Λ' up to dimension 12. The exact bound for Q_b on 2-elementary quotients has been shown to be equal to $\frac{6\cdot 7}{16} = \frac{21}{8} = 2.625$ in dimension 10. This is a simple matter of classifying binary codes of weight $w > 4$. This classification is easily extended in dimensions 11 and 12. It turns out that the highest values for $Q_b(\Lambda)$ on the set of lattices with 2-elementary quotients Λ/Λ' are obtained by lifting unique even codes C_{11} and C_{12} of weight 6. We first define C_{12} by the generator matrix

$$G_{12} = \begin{pmatrix} 1&1&1&1&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&1&1&1&1&0&0&0 \\ 1&1&0&1&0&0&1&0&0&1&1&0 \\ 1&0&0&1&1&0&0&1&0&1&0&1 \end{pmatrix},$$

then C_{11} by the generator matrix G_{11} obtained from G_{12} by deleting the last column and the last row. The weight distribution of C_{11} is $6^6 \cdot 8$ and that of C_{12} is $6^{12} \cdot 8^3$, which gives Q_b the lower bounds $\frac{27}{8} = 3.375$ and $\frac{81}{16} = 5.0625$, respectively, reasonably close to van der Waerden's bounds (4.768... and 5.960..., respectively).

I conjecture that 2*-elementary quotients still produce the largest possible values for* $\mathbb{Q}_b$ *in dimensions* 10*,* 11 *and* 12*.*

References

[K-M-S] W. Keller, J. Martinet, A. Schürmann, *On classifying Minkowskian sublattices*, Math. Comp. **81** (2012), 1063-1092; preprint at arXiv:0904.3110v3.

[M] J. Martinet, *Perfect Lattices in Euclidean Spaces*, Grundlehren **327**, Springer-Verlag, Heidelberg (2003).

[M1] J. Martinet. *Sur l'indice d'un sous-réseau* in "Réseaux euclidiens, designs sphériques et formes modulaires", J. Martinet ed., Monographie Ens. Math. **37**, Genève (2001), 165–213.

[M-S] J. Martinet, A. Schürmann, *Bases of minimal vectors in Euclidean lattices, III* Inter. J. Number T. **08, 2** (2012), 551–567; preprint at arXiv:1105.5889v1

[Ry] S. S. Ryshkov, *On the problem of the determination of quadratic forms in many variables*, Proc. Steklov Inst. Math. **142** (1979), 233–259 ; Russian original: 1976.

[vdW] B.L. van der Waerden, *Die Reduktionstheorie der positiven quadratischen Formen*, Acta Math. **96** (1956), 265–309.

[Wa] G.L. Watson, *On the minimum points of a positive quadratic form*, Mathematika **18** (1971), 60–70.

[Za] N.V. Zahareva, *Centerings of* 8*-dimensional lattices that preserve a frame of successive minima*, Proc. Steklov Inst. Math. **152** (1982), 107–134. (Russian original: 1980.)

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