LATTICES MODULO 2 AND 3 (COMMENTS)

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Abstract. We write a few complements (and an erratum) to the two papers *Reduction Modulo 2 and 3 of Euclidean Lattices*, published in J. Algebra (2002) and Contemporary Math. (2004), cited [LM1] and [LM2] below, respectively.

1. Erratum

In an e-mail dated May 14th, 2012, R. Idel pointed out to me that in Theorem 1.3 of [LM1], one must assume that $d = 2$ or 3. Fortunately larger values of d are not considered in the paper.
Also, in (1), "orthogonal" holds if and only if $N(y) = N(x)\,(= 2m)$.

Table III, line 1, *read* $s_4 = 45$.

2. Tables of [LM1]

2.1. **Perfect lattices in dimension 8.** Since [LM1] was written, Dutour-Sikirić, Schürmann and Vallentin proved that the list of perfect lattices listed in my homepage is indeed the complete list. In my homepage, they are now organized as a collection of six lists p8di, $i = 7, 6, 5, 4, 3, 2$ where i stands for the dimension of a largest perfect cross-section having the same minimum; p8d7 is a re-ordering of the former list of *Laïhem lattices* $lh(j), j = 1, ..., 1175$.

Two of these lattices are listed in Table 2.11: $\mathbb{E}_8 = p8d7[1]$ and $L_8^4 = p8d7[4]$. The others are

$$p8d7[181]\ (= lh(179)),\ p8d5[126]\ (= batu(5)),\ \text{and}\ p8d5[128]\ (= nap(160))\,.$$

They could have been added to Table 2.12. For the first one, we have $s_6 = 40, s_8 = 45, s_{10} = 96, s_{12} = 158$, whereas for $p8d5[126]$ (resp. $p8d5[128]$) we need the consideration of all even norms between the minimum $m = 14$ (resp. $m = 12$) and $2m$.

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2.2. **Dimensions 7 and 8**(June 10th, 2016). There is not a lot to say on Table 2.10, since all lattices lying in a small enough neighbourhood of any of the lattices listed there could be added to it, including all lattices of the half-open Voronoi path $(\mathbb{E}_6$—$\mathbb{E}_6^*]$.

I have then considered the Voronoi paths in dimension 7 of the form L—L' where at least one of the lattices L, L' is one of the three perfect lattices for which norm $2m$ suffices, namely P_7^5, P_7^{20} and P_7^{23}, and tested whether $2m$ would suffice on a neighbourhood of the convenient end point(s). This occurred for 10 paths, namely 3 paths P_7^1—P_7^5 out of 10, 5 paths P_7^5—P_7^5 out of 6, and the unique paths P_7^5—P_7^{20} and P_7^5—P_7^{23}. Actually I can prove more: the norms between m and $2m$ suffice for the whole last seven paths (those which only involve P_7^5, P_7^{20} and P_7^{23}); for the first three, this holds only on an interval $[t_0, 1]$ with $0 < t_0 < 1$.

This shows that $2m$ suffices in dimension 7 on some subsets of dimension 1 (even possibly of a larger dimension), whereas only isolated examples had been found in [LM1] and [LM2].

I have then turned to dimension 8, restricting myself to the perfect lattices $p8d5[126]$ ($s = 36$) and $p8d5[128]$ ($s = 37$), for which the small value of s makes easy the construction of Voronoi neighbours. I have detected two Voronoi graphs of the form $p8d5[126]$–$\mathbb{E}_8$ and $p8d5[128]$–$\mathbb{E}_8$ on which all lattices possess representatives modulo 2 of norm $\le 2m$. I also found a path $p8d5[128]$–$\mathbb{E}_8$ on which the bound $2m$ applies uniquely for $t = 0$ and $\frac{1}{3} \le t \le 1$. This suggests that for many of the numerous paths of the form $\mathbb{E}_8$–L the bound $2m$ suffices near $\mathbb{E}_8$. Another example will be given in Subsection 2.3.

I state the results above as a proposition.

Proposition 2.1. *For all dimensions $n \le 8$ the space of lattices up to similarity contain subsets of dimension $d \ge 1$ in which all lattices possess representatives modulo 2 of norm not larger than twice their minimum.* □

Whether examples of dimension $n(n+1)/2$ exist beyond dimension 6 remains an open problem, as well as the existence of non-isolated examples beyond dimension 8.

2.3. **Extension of the Tables of [LM1]** (extended on June 12th, 2016). An interesting example for Table 2.11 has been missed, namely the 5-modular lattice L of dimension 8 and minimum 4, strongly eutactic of perfection co-rank 1, the lattice $stg60$ of the file *strongly eutactic lattices* in the *catalogue of Perfect Lattices* of my homepage. Thus the following line could have been added to Table 2.11:

$n = 8 \quad \ell = 4 \quad \Lambda_{8,5-mod} \quad s_4 = 60 \quad s_6 = 120 \quad s_8 = 300\,.$

Here are two other definitions for L.

(a) Algebraic. $L = (\mathfrak{M}, \mathrm{Trd}_{H/\mathbb{Q}}(x\overline{y}))$, where H is the quaternion fields with center $\mathbb{Q}(\sqrt{5})$ ramified exactly at the two real primes of its center, $\mathfrak{M}$ is a maximal order in H, and Trd is the reduced trace. Its automorphism group, of order $28800 = 2^7 \cdot 3^2 \cdot 5^2$, acts transitively on each of his layers of norm $4, 6, 8$.

(b) Craig. The cross-section of Craig's lattice $\mathbb{A}_9^{(2)}$ (alias Barnes's P_9) by the hyperplane orthogonal to the unique pair of minimal vectors of its dual is isometric to L.

Since the perfection co-rank of L is equal to 1, L belongs to a Voronoi path $L(t)$, indeed connecting two copies of $\mathbb{E}_8$. A matrix representation for such a path is $U(t) = A + t\,M$, of minimum 2, where

$$A = \begin{pmatrix} 2 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 2 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

It turns out that *all lattices $L(t)$, $0 \le t \le 1$, have representatives modulo* 2 *in the interval* $[2, 4]$. The lattice L is a scaled copy of $L(\frac{1}{2})$, and is well defined as *the* eutactic lattice of the path $L(t)$. This family of lattices has moreover the following two properties:

(1) For every $t \in [0, 1]$, $L(1-t)$ is isometric to $L(t)$;
(2) For every $t \in [0, 1]$, $L(t)^*$ is similar to $L(t)$.

This results from the two formulae

$$U(1-t) = {}^tPU(t)P \quad \text{and} \quad {}^tQ\big((-t^2+t+1)U(t)^{-1}\big)Q = U(t)\,,$$

where

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This Voronoi path is an analogue of the Voronoi path $\mathbb{D}_4$—$\mathbb{D}_4$, having as mid point the 4-modular, strongly eutactic lattice $\mathbb{A}_2 \otimes \mathbb{A}_2$, all lattices of which also have representatives modulo 2 in the interval $[2, 4]$.

3. Odd Lattices (Section 4 of [LM2])

Theorem 4.2 of [LM2] shows that in an integral lattice of minimum m all layers up to norm $2m+1$ must be considered when listing shortest representatives of classes modulo 2. This theorem gives a precise description of classes of norm $2m+1$ when m is odd, and is then applied to $\mathbb{E}_7^*$ scaled to minimum 3.

(February 21st,2022) An example with m even is provided by the lattice $\mathbb{D}_{10}^+$ scaled to minimum 4. One must then consider the layers of norms 4, 5, 8 and 9, and the weighted formula reads

$$90 + 256 + \Big(\frac{10}{10} + \frac{1680}{4}\Big) + \frac{2560}{10} = 2^{10} - 1\,.$$

On the basis (ε_i) for $\mathbb{Z}^{10}$ rescaled to $N(\varepsilon_i) = 2$, vectors of norm 9 congruent modulo 2 are permutations of $\frac{-3\varepsilon_1 \pm \varepsilon_2 \pm \cdots \pm \varepsilon_{10}}{2}$ (with an even number of minus signs), of norm $2 \times \dfrac{3^2 + 9 \cdot 1^2}{4} = 9$, and we have

$$\frac{-3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots + \varepsilon_{10}}{2} \cdot \frac{\varepsilon_1 - 3\varepsilon_2 + \varepsilon_3 + \cdots + \varepsilon_{10}}{2} = 2 \times \frac{-6+8}{4} = 1\,,$$

as predicted by Theorem 4.2 of [LM2].

4. On Lemma 3.2 of [LM2]

(Completed February 21st, 2022) This lemma states that an integral lattice of minimum 4 which is the even sublattices of an integral lattice of minimum 3 contains classes modulo 2 of minimal norm 12. The converse is not true: given a class of minimal norm 12 represented by a vector X and a Gram matrix A for Λ, $\frac{1}{2}X$ defines an *integral* lattice of minimum 3 if and only if $AX \equiv 0 \mod 2$. Such an example is provided by the lattice K'_{15}; see below.

Inside the Leech lattice, the series Λ_n has been dealt with in [LM2]. Representatives of norm 8 exist for $n \le 6$ and $n = 8, 9, 10, 24$ and for no other dimensions. These lattices are even sublattices of integral lattices of minimum 3 for $\Lambda_7 \sim \mathbb{E}_7$, $\Lambda_{11}^{\mathrm{max}}$, $\Lambda_{12}^{\mathrm{mid}}$, $\Lambda_{12}^{\mathrm{max}}$, and all Λ_n for $13 \le n \le 23$. There remains the cases of $\Lambda_{11}^{\mathrm{min}}$ and Λ_{12}^{min}, for which the four norms $4, 6, 8, 10$ are needed.

Besides the series Λ_n, two important other series exist inside the Leech lattice Λ_{24}, that we are going to investigate:
- the series K_n for $7 \le n \le 17$ ($K_n = \Lambda_n$ if $n \le 6$ or $n \ge 18$);
- the series K'_n for $3 \le n \le 10$ and $14 \le n \le 21$
  ($K'_n = K_n$ if $n = 11, 12, 13$; $K'_n = \Lambda_n$ if $n \le 2$ or $n \ge 22$),

that we consider below for $n \le 15$.

We know from [LM1] that representatives of norm ≤ 8 exist for K_7, K_8 and K_{12}. For K_n, $n = 9, 10, 11$, the smallest norms are $4, 6, 8, 10$. Finally, K_{13}, K_{14}, K_{15} requer norms 4 to 12, and are the even sublattices of lattices L_{13}, L_{14}, L_{15} of minimum 3.

Norms 4 and 6 suffice for K'_3, K'_4, K'_5, K'_6, then norms up to 8 for K'_8, up to 10 for K'_7 and K'_{14}, and norms up to 12 are needed for K'_9, K'_{10} and K'_{15}; K'_9 and K'_{10} are even sublattices of integral lattices of minimum 3, but not K'_{15}, for which the numbers of vectors of each of these norms satisfy the formula

$$822 + 10116 + 15687 + 6140 + 2 = 32767 = 2^{15} - 1\,,$$

which shows that norm 12 is needed.

The Kneser neighbour of K_{12} with respect to a norm-4 vector has a decomposition $\mathbb{Z} \perp 3\mathbb{Z} \perp K'_{10odd}$, where K'_{10odd} is a 3-modular, strongly eutactic lattice of minimum 3, with $s = 40$; its even sublattice is isometric to K'_{10}. By cross-section, one obtains a lattice K'_{9odd} with even sublattice K'_9.

References

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