Preface to the English Edition

This book discusses a beautiful and central problem in mathematics, which involves geometry, number theory, coding theory and group theory. The geometrical objects we consider are *lattices*, i.e. discrete subgroups of maximal rank in a Euclidean space. To such an object we attach its canonical *sphere packing*, namely the set of (non-overlapping) spheres centred at all points of the lattice whose common radius is half the minimal distance of two lattice points. Assuming some regularity conditions, a sphere packing has a density. The question of estimating the highest possible density of a sphere packing in a given dimension n is a fascinating and difficult problem: the answer is known only up to dimension 3, and the case of dimension 3 was settled very recently by Hales, who gave a positive answer to an old conjecture of Kepler. The case of lattice packings is slightly easier, though still highly non-trivial: in this case, the answer is known up to dimension 8, thanks to the difficult 1935 paper [Bl2] of Blichfeldt.

The book is centred on the study of *extreme lattices*, those on which the density of the canonical sphere packing attains a local maximum, and various related questions. This is based on the French version published by Masson in 1996 under the title "Les réseaux parfaits des espaces euclidiens". However, it is very far from being a word-for-word translation. Every chapter has been rewritten, some completely, For example, I found a much simpler proof of the classification of 5-dimensional perfect lattices after the French version was printed, and as a result Section 4 of Chapter 6 has been completely changed.

A more detailed list of the major changes can be found in the *footnotes* to the *Introduction* (which otherwise follows the French version). The reader who possesses the French Edition can find an erratum and a full list of the changes on the web page http://www.math.u-bordeaux.fr/~martinet/. A number of readers of the French version supplied lists of corrections and of course these have all been made in the new edition. I should like to thank more specially Anne-Marie Bergé, Philippe Calame and Maurice Mischler for their comments.

During the six years since the French version was published there has been important progress in some areas directly connected with the main topics of the book. For this reason an essentially descriptive appendix has been added, whose main objective is to discuss the work of B.B. Venkov and others connecting the local theory of lattices with *spherical designs*.

I must mention here the recent work of Bavard, who has incorporated the theory of Chapter 10 (which considers families of lattices which form orbits under the action of a Lie subgroup of the linear group) in a more general setting in the context of Riemannian geometry. This has provided unified definitions for some ad hoc notions that were introduced in order to obtain reasonable finiteness theorems or conditions to guarantee that certain lattices are algebraic. However, it would have taken too much space to include a full discussion of this work here, and I can only refer the reader to Bavard's papers mentioned in the bibliography.

Finally, the bibliography has been updated and the index has been greatly enlarged. There is an extensive list of symbols.

The following are the most important changes that have been made (besides the modifications to the appendices and the bibliography):

- Chapter 1: Section 1.9 contains further material on integral lattices.
- Chapter 4: Changes in Section 4.8, and six new exercises.
- Chapter 6: Section 6.4 has been completely rewritten.
- Chapter 8: Section 8.1 has been completely rewritten to take into account the modifications to the appendices.
- Chapter 9: There is a new Section 9.7 devoted to recent results by Batut and by Bavard.
- Chapter 14: The first section has been shortened, and a new theorem has been incorporated in Section 14.6.

Many improvements were suggested to me by Neil Sloane after he read in detail my manuscript. I thank him for his important contribution to the book. I would like to thank also Henri Cohen for his help in using Springer's Latex.

Finally, I express my heartfelt thanks to my wife Titou for her patience during the years I have been writing the two editions of this book.

Talence, March 28th, 2002.

Jacques Martinet

General Principles for the Notation

The following notation will be used throughout the book unless explicitly stated to the contrary. (For instance, it may happen that we consider a sequence $L_1, L_2, \ldots$ of lattices.)

1. E denotes a Euclidean space, whose dimension is denoted by n.
2. The notation $F \perp F' \perp \ldots$ is used for *orthogonal* direct sums.
3. In a Euclidean space F of dimension $n+1$ (resp. n), $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ (resp. $(\varepsilon_1, \ldots, \varepsilon_n)$) denotes an orthonormal basis for F. Latin letters, usually $(e_1, \ldots, e_n)$, are used for a basis which is not *a priori* orthonormal.
4. The symbols Λ, Λ', L, L', M etc. denote lattices. A subscript such as Λ_m indicates the dimension.

The notation for duality deserves special comments. The symbol M^* means first that M is a lattice, and that we consider its dual *over* $\mathbb{Z}$, even if M is endowed with some other algebraic structure. If we need to consider the dual of M over some ring containing strictly $\mathbb{Z}$, we write $M^\sharp$. Finally the set of invertible elements in a ring R is denoted by $R^\times$.

The notation from linear algebra is essentially the standard one ($\mathrm{End}(V)$, $\mathrm{GL}(V)$, $\mathrm{SL}(V)$, $\mathrm{O}(V)$). As for matrices, we denote by $\mathcal{M}_{p,q}(R)$ the module of matrices with p rows and q columns over a ring R, and use the shorter notation $\mathcal{M}_n(R)$ for $\mathcal{M}_{n,n}(R)$.

Finally, we use two notions of norm. In a Euclidean space the norm $x \cdot x$ is denoted by $N(x)$. In an algebra L/K the determinant of the endomorphism $y \mapsto xy$ is denoted by $\mathrm{N}(x)$ (or $\mathrm{N}_{L/K}(x)$). Note that $N(x) = \|x\|^2$ is the square of the classical Euclidean norm.

Statements (theorems, propositions, definitions, tables, etc.) are numbered according to the scheme **Theorem a.b.c.**, **Table a.b.c.**, etc., where **a** is the number of the chapter, **b** is that of the section, and **c** denotes the **c**-th statement in Section **b** of Chapter **a**, independently of its nature. A reference to Theorem 5 of Section 3 in Chapter 4 is thus written in the form Theorem 4.3.5.

Introduction

This is based on the Introduction to the French version, the main differences being the addition of footnotes to indicate places where significant changes have been made in the English version.

Throughout this book, E denotes an n-dimensional *Euclidean space*, i.e. a finite-dimensional real vector space endowed with a positive definite symmetric bilinear form, namely the *scalar product on* E. A *lattice in* E is a discrete finitely generated subgroup of E of maximal rank, i.e. of rank n.

Let Λ be a lattice, and let d be the smallest distance to the origin of the other points of Λ; d is also the smallest distance between two points of Λ. Consequently, balls of radius $R = \frac{d}{2}$ centred at points of Λ do not overlap (their intersection contains at most one point). We say that they are *packed* by Λ. The *density* of this packing is an important invariant of Λ, sometimes called by abuse of language "the density of Λ". Another important invariant is the *kissing number of* Λ, the number of length d vectors in Λ (the *minimal vectors of* Λ), which we will usually denote by $2s$ (these vectors occur in pairs $\pm x$).

Lattices for which this density is a local maximum, i.e. such that this density does not increase when one performs a sufficiently small deformation, were called *extreme lattices* by Korkine and Zolotareff. Their goal, which they were able to carry out up to dimension 5, was to classify extreme lattices in a given dimension and then to extract from this classification a list of the absolutely densest ones, which they called the *absolutely extreme* lattices. However, we prefer to call these *critical lattices*, in conformity with the tradition in the geometry of numbers.

The notion of a *perfect lattice* is a less restrictive one, which can be expressed within the framework of linear algebra, whereas inequalities are required to characterize extreme lattices. These are the lattices with the property that the set of projections onto the lines containing the minimal vectors spans the space of symmetric endomorphisms of E. Less formal properties will show up later, for instance the following, which goes back to Voronoi: a lattice Λ is perfect if the image of Λ under any sufficiently small deformation which is not a similarity of E contains fewer minimal vectors than Λ itself. The formal definition shows that the inequality $s \geq \dfrac{n(n+1)}{2}$ holds for all perfect lattices. Let us also quote Voronoi's finiteness theorem: up to similarity, there exist only finitely many perfect lattices in any given dimension.

*

* *

In fact, the theory of perfect lattices and sphere packings was developed up to the 1960s as part of the theory of positive definite real quadratic forms. However, we prefer to work with the *Hermite invariant* of a lattice Λ rather than with its density. This is defined by

$$\gamma(\Lambda) = N(\Lambda)\ \det(\Lambda)^{-1/n}$$

where $N(\Lambda)$ (the *norm* or *minimum of* Λ) is the minimum on $\Lambda \smallsetminus \{0\}$ of the scalar products $N(x) = x.x$, and $\det(\Lambda)$, the *determinant of* Λ, is the determinant of the matrix of pairwise scalar products of the vectors of any $\mathbb{Z}$-basis $\mathcal{B}$ for Λ (the *Gram matrix of* $\mathcal{B}$). Then the density is proportional to $\gamma(\Lambda)^{n/2}$. Since the density is bounded from above (by 1), one may consider the upper bound of $\gamma(\Lambda)$ on the set of all n-dimensional lattices: this is the *Hermite constant for dimension* n, denoted by γ_n. (A more natural invariant would be γ^n, which was shown by Korkine and Zolotareff to take rational values on perfect lattices.)

The definitions above, which were expressed in terms of lattices, are easily translated into the language of *quadratic forms over* $\mathbb{R}^n$ (homogeneous degree 2 polynomials in n variables $x_1, \dots, x_n$): with a given a pair $(\Lambda, \mathcal{B})$ where Λ is a lattice and $\mathcal{B} = (e_1, \dots, e_n)$ is a basis for Λ over $\mathbb{Z}$, we associate the quadratic form

$$q(x_1, \dots, x_n) = N(x_1 e_1 + \cdots + x_n e_n)\,.$$

To replace $\mathcal{B}$ by another basis amounts to replacing q by an *equivalent form* (equivalent under a transformation of $\mathrm{GL}_n(\mathbb{Z})$). Passing to quotients, we establish a one-to-one correspondence between isometry classes of lattices and equivalence classes of quadratic forms. This dictionary relating lattices and quadratic forms is studied in detail in Chapter 1; notice that it induces a one-to-one correspondence between similarity classes of lattices, the natural object in the theory of perfect or extreme lattices, and classes of quadratic forms *up to proportionality*.

When working with quadratic forms, the existence of the constant γ_n is not evident. It was established by Hermite in a letter to Jacobi (dated August 6, 1845), although Lagrange (resp. Gauss) had already calculated it for dimension 2 (resp. 3).

After Hermite's work, the problem of determining γ_n beyond dimension 3 was considered. The case of dimensions 4 and 5 was solved by Korkine and Zolotareff in a series of three papers published in *Mathematische Annalen* between 1872 and 1877, in which they indeed determine *all* perfect forms in dimension $n \leq 5$ (although without using the word "perfect", which was introduced by Voronoi some thirty years later).

The translation in terms of lattices was done by Minkowski, who realized that *estimating the minima of the set of all real (positive definite) quadratic forms on* one *particular lattice* (for instance on $\mathbb{Z}^n$, which is the aim of the classical theory) is essentially the same problem as *estimating the minima on the set of all lattices in* $\mathbb{R}^n$ *on* one *particular form* (for instance, those of $x_1^2 + \cdots + x_n^2$, which defines the canonical Euclidean structure on $\mathbb{R}^n$).

This extremely original idea, of fundamental importance despite its simplicity, constitutes the birth certificate of the geometry of numbers, a new branch of mathematics whose autonomous existence can be reasonably dated to 1896, the year Minkowski's book *Geometrie der Zahlen* appeared.

As for the Hermite constant, Minkowski's geometrical methods prove the existence of upper bounds which are linear in n, whereas the arguments that Hermite used to prove existence give only exponential bounds.

Minkowski never considered perfect forms, and the next step was made by Voronoi; it consists of three articles published in Crelle's journal between 1907 and 1909. It is his first article which chiefly interests us: Voronoi proves here his finiteness theorem referred to above, characterizes the extreme forms among perfect forms (they must have the additional property of being *eutactic*), and develops an algorithm with which he recovers the classification results found thirty years earlier by Korkine and Zolotareff.

Unfortunately, Voronoi died in 1908, leaving his research unfinished. He was clearly working on the case of dimension 6, where he had found the first non-eutactic perfect form, but one had to wait half a century for the completion of the 6-dimensional classification, obtained by Barnes in 1957. Barnes also discovered many other perfect lattices, including the lattice K_{11}, and together with G.E. Wall also found Λ_{15} and Λ_{16}. These are the densest lattices known in dimensions 11, 15 and 16 and are widely believed to be the critical lattices in these dimensions.

We must mention two important results that were obtained in the period between the works of Voronoi and Barnes. The first is the determination by Blichfeldt in 1935 of the Hermite constant in dimensions 6, 7 and 8 (and the reduction by Mordell in 1944 of the calculation of γ_8 to that of γ_7); the other is Coxeter's 1951 paper on root lattices and their relatives, and the discovery in a joint work with Todd of the K_{12} lattice, which they showed to be extreme, and which is very likely the densest 12-dimensional lattice.

After Barnes, among various work devoted to local methods, I would like to mention the following:

• Watson's work, extending the methods of Korkine and Zolotareff in order to attempt to classify perfect lattices in dimensions 6 and 7.

• Kaye Stacey's work (1975), which obtained an essentially correct list of the 7-dimensional perfect lattices. However, she was unable to establish that her list was complete, for lack of an efficient identification algorithm; her results were confirmed by Jaquet in 1990. Barnes and Jaquet worked with the Voronoi algorithm, whereas Stacey used the of Korkine–Zolotareff–Watson methods.

• The construction of important lattices, by Leech (in particular, the fundamental 24-dimensional lattice which bears his name), by Conway and Sloane (for instance, the "laminated lattices" Λ_n), by Quebbemann in dimension 32, ... , all of which are perfect.

• The systematic study by Conway and Sloane of perfect lattices up to dimension 7.

• The definition by Bergé and Martinet of new notions of perfection and extremality restricted to special classes of lattices (e.g., lattices which are extreme with respect to a given automorphism group or with a given section, or as isodual lattices, or dual-extreme lattices – an analogue of the Hermite invariant for which a lattice and its dual play a joint rôle).

*

*   *

We begin with a chapter on general properties of lattices. The second chapter on inequalities is more technical and is not used much before Chapter 6. The story really begins in Chapter 3,[1] where we introduce the notions of extremality, perfection and eutaxy of a lattice, and the analogous notions involving the pair consisting of a lattice and its dual. It is thus quite possible to start reading the book at the third chapter. This chapter is a particular case of a more general theory to be developed in Chapter 10, where we shall consider families of lattices which are orbits under the action of a closed subgroup of the general linear group. However, the basic techniques developed in Chapter 3 will play a fundamental rôle, and a detailed study of the classical situation may prove useful for understanding Chapter 10.

Chapter 4 is devoted to root lattices, whose rôle in numerous domains of mathematics (group theory, Lie algebras, etc.) is well-known. It is followed by a fifth chapter dealing with various lattices which are easily constructed as modifications of root lattices. Both these chapters contain many applications of the theory developed in Chapter 3. Moreover, the families of lattices that we construct in these two chapters contain almost all perfect lattices up to dimension 6. (The complete list is obtained by considering two extra families of lattices, which we construct in Chapter 8.)

Chapter 6 is devoted to the classification of perfect lattices. We give complete proofs up to dimension 5, but content ourselves with a description of the known results in dimensions 6 and 7 and a few indications for dimension 8, since classification is not known from $n = 8$ onwards. The proofs we give in this chapter originate from the work of Korkine and Zolotareff, although we must emphasize that they proved more than they stated: for instance, the

[1] We also introduce here the notion (not mentioned in the French edition) of a strongly eutactic lattice, which plays an important rôle in the study of spherical designs.

necessary condition $s(\Lambda) \geq \frac{n(n+1)}{2}$ suffices to ensure perfection (and even extremality) up to dimension 5, except for 5-dimensional lattices possessing a $\mathbb{D}_4$-section with the same norm, where one must assume the existence of five directions of minimal vectors outside the section.[2] We also describe all dual-extreme lattices up to dimension 4 (the classification is not known from $n = 5$ onwards), but we do not give the complicated proof of the classification in dimension 4.

Chapter 7 is devoted to the classical Voronoi algorithm. Now we drop the language of lattices for that of quadratic forms, which is here more convenient. (However, the lattice point of view shows up in certain places, and Voronoi himself sometimes used it without saying so.) We describe Voronoi's procedure, which allows one to attach to every perfect form (defined up to proportionality by giving it a fixed minimum on $\mathbb{Z}^n$) a polyhedral convex cone, its *Voronoi domain*, and to every facet of this cone a new perfect form, which is *contiguous* to the original one (through the given facet), giving in this way the set of equivalence classes of perfect forms a structure of a finite, connected graph. Following Voronoi, we use this method to recover the classification of perfect forms of dimension $n \leq 5$ that we established in the preceding chapter.

The reader *could* learn the Voronoi algorithm by directly reading Chapter 13, ignoring the first four sections of Chapter 7. However, the remark we made above about Chapters 3 and 10 applies to Chapters 7 and 13. Note also that reading Chapters 6 and 7 directly after Chapter 3 (in any order) is also possible for a reader having some acquaintance with the zoology of lattices constructed in Chapters 4 and 5.

Chapter 8 is a continuation of the constructions of lattices given in Chapters 4 and 5 that were interrupted by two chapters on classification problems. We consider here constructions of a more algebraic nature, making use of orders in semi-simple algebras, essentially in fields with complex multiplication or quaternion fields. This chapter contains various original results. The constructions of Barnes and Coxeter–Todd, which make use of the ring of Eisenstein integers, are described and generalized in Sections 8.4 and 8.5, after some analogous constructions making use of the Hurwitz order have been performed in Sections 8.2 and 8.3. This is followed by sections describing new constructions relying on the structure of some left ideals in quaternion skew-fields whose centres are no longer the field of rational numbers. Numerous classical lattices are endowed in this way with various algebraic structures, and new lattices, often unimodular, are obtained; a résumé of the original results of this chapter appeared in the proceedings of the Paris Number Theory Seminar of 1992–1993.

Chapter 9, which is based on results by A.-M. Bergé and myself, deals with classifications of lattices (or of pairs of a lattice and its dual) according to

[2] Here is one important difference from the French edition: following [Mar5], we have written much simpler proofs, relying on methods of Watson.

properties of their minimal vectors. As an application, we prove a finiteness theorem which contains results of Voronoi (on perfect lattices) and Avner Ash (on eutactic lattices), and also a dual version due to A.-M. Bergé of Voronoi's finiteness theorem. This subdivision of lattices into finitely many *minimal classes*, which contains the classification of perfect lattices (those for which the minimal class reduces to their similarity class), is known only for $n \leq 5$ (resp. only for $n \leq 3$ in case of pairs (Λ, Λ^*)). We present the proofs in dimensions $n \leq 4$ (resp. $n \leq 3$). Some results of Watson about the kissing number have a natural interpretation in terms of minimal classes; it would be interesting to extend at least partially these classification results beyond dimension 5. (Minimal classes are kind of "orbifold"; they correspond to the cell decomposition in the space of positive definite quadratic forms; this cell decomposition has been considered by the Russian school, notably by Štogrin, Baranovskii, and Ryshkov.)

Chapter 10 gives a broad generalization[3] of Voronoi's characterization of extreme lattices as those which are both perfect and eutactic. Notions of perfection and eutaxy for a subspace $\mathcal{T}$ of the space $\mathrm{End}^s(E)$ of all symmetric endomorphisms were defined in Chapter 3. We consider here families of lattices that constitute a homogeneous space under the action of a closed (possibly connected) subgroup $\mathcal{G}$ of $\mathrm{GL}(E)$, which we also assume to be invariant under transposition. Being a Lie subgroup of $\mathrm{GL}(E)$, $\mathcal{G}$ possesses a tangent space at the identity, and Voronoi's theory generalizes, working in the symmetrized set $\mathcal{T} \subset \mathrm{End}^s(E)$ of this tangent space.

The preceding theory is applied in Chapter 11 to two examples of a great practical importance.

The first one, studied in the first four sections, is that of G-lattices, the set of lattices in E whose automorphism group contain a given finite subgroup G of the orthogonal group $\mathrm{O}(E)$. The idea of considering such lattices originates in algebraic number theory, G being a Galois group acting on units modulo torsion.

Our second example, studied in the remainder of the chapter, is that of lattices endowed with a given isometry σ onto their duals. The case of symplectic lattices (i.e., such that $\sigma^2 = -\mathrm{Id}$) is of particular importance in the theory of complex Abelian varieties.

In Chapter 12, we consider the following two questions: How can one classify the sections of a given lattice? And how can one characterize the perfection and eutaxy properties of a lattice for which we know *a priori* a section?

These problems somewhat resemble those which were examined in Chapter 10, and it might well be possible to unify the two theories by considering

[3] Notions of perfection and eutaxy, which generalize those we consider in this book, have been developed recently by Bavard within the framework of Riemannian geometry.

lattices which are parametrized by submanifolds with boundary in the vector space of endomorphisms. No such theory is yet available.

In Chapter 13 – the last largely theoretical one – we consider extensions of the classical classical Voronoi algorithm that was studied in Chapter 7. The theory is applied relative to a subspace $\mathcal{T}$ of $\mathrm{End}^s(E)$, but is in practice useless unless the contiguity algorithm stabilizes an interesting class of lattices. This is the case with G-lattices in the sense of Chapter 11: $\mathcal{T}$ is the subspace consisting of elements of $\mathrm{End}^s(E)$ which commute with G and the operation of "$\mathcal{T}$-contiguity" transforms a G-lattice into a new G-lattice; this property also holds for the family of lattices possessing a given section with the same minimum. Unfortunately, we are able to generalize the Voronoi algorithm only to families which constitute a cone[4] inside an affine subspace of $\mathrm{End}^s(E)$. To discover an algorithm for families such as those of symplectic lattices would be of a great interest for the solution of classification problems.

We have collected in the final chapter, Chapter 14, some numerical data (Gram matrices of perfect or eutactic lattices, tables of invariants of some remarkable lattices). Numerical complements are also available on the WEB; see the Batut–Martinet Catalogue of Perfect Lattices

http://www.math.u-bordeaux.fr/~martinet/

and the Nebe–Sloane database

http://www.research.att.com/~njas/lattices/;

see also the tables in Chapter 6.

The book ends with two appendices, the first one of an algebraic nature written on request of certain colleagues who wished a guide to algebraic constructions of lattices. It is certainly not an accident if so many "beautiful" lattices, especially in even dimensions, and above all in dimensions divisible by 4, possess rich algebraic structures, and the search for algebraic constructions is indeed still an active domain in lattice theory. The second appendix is a short account of the connections which exist between the theory of lattices and that of spherical designs.[5]

*
*   *

Every chapter except the first and the last contains numerous exercises, some of which are not in the French edition. Their numbers vary from chapter to chapter. In general they can be solved by hand, but in a few cases computer programs (especially those for finding minimal vectors) can be used to avoid

[4] Recently, Bavard has considered in [Bav3] a generalization which deals with symplectic lattices in the Siegel space parametrized by the Poincaré upper half-plane.

[5] This appendix is new; the first appendix is a contraction of the four appendices of the French version. This is partially compensated for by an expansion of Section 8.1.

tedious calculations. For the sake of simplicity, those chapters involving heavy computational methods contain only few exercises.

Each chapter except the last (and the appendices), ends with "notes", where we quote some results which could not be incorporated within the chapter. The notes also include a number of historical remarks. They have been extended to include results discovered after the French edition was written.

*
* *

This book is devoted to the study of *local methods* in lattice theory. I have chosen to centre it on the perfection property. I realized while writing it that eutaxy, which appears at first sight to be just a minor restriction, is just as important as perfection. Moreover, the discovery of the existence of an amazing number of 8-dimensional perfect lattices[6] in comparison with smaller dimensions (3, 7, and 33 in dimensions 5, 6, and 7), strongly suggests that the eutaxy property could be used for limiting the number of lattices that must be considered when classifying extreme lattices. For this reason, the eutaxy property also plays a major rôle in this book.

Venkov has recently discovered that the theory of modular forms can be used to prove *a priori* that certain lattices are extreme, or at least eutactic; the proofs rely on the theory of *spherical designs*. We shall not give a detailed study of this theory, about which nothing had been written at the time the French edition was printed. However, an appendix, of a purely descriptive nature, has been added to the English edition; the proofs can be read in the recent article [Ven3].

Another recent breakthrough was made independently by Elkies and Shioda, who constructed dense lattices using algebraic geometry. They obtain integral lattices for which they can determine the minimum and the determinant, and even the kissing number. However, up to now, this theory has not yielded results concerning perfection nor eutaxy.

*
* *

The notion of an integral lattice, although it appears here and there in this book (perfect lattices are proportional to integral lattices, as one knows since the time of Korkine, Zolotareff, and Voronoi), was not at the heart of our study. As a consequence, the content of this book is essentially disjoint from that of Conway and Sloane's "bible" (*Sphere Packings, Lattices and Groups* [C-S], [C-S'], [C-S"]), which remains a necessary tool for any one who wishes to study lattices.

[6] 10916 were known at the time this introduction was translated

Contents