ON THE INDEX SYSTEM OF WELL-ROUNDED LATTICES

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Abstract. Let Λ be a lattice in an n-dimensional Euclidean space E and let Λ' be a Minkowskian sublattice of Λ, that is, a sublattice having a basis made of representatives for the Minkowski successive minima of Λ. We consider the set of possible quotients Λ/Λ' which may exists in a given dimension or among not too large values of the index $[\Lambda : \Lambda']$, indeed $[\Lambda : \Lambda'] \leq 4$, or dimension $n \leq 8$.

1. Introduction

Extending a deformation argument used in [M2] to prove Minkowski's theorem on successive minima (Theorem 2.6.8; 1996 in the French edition), I proved in [M1] that the sets of isomorphisms classes of quotients Λ/Λ' for Λ' a Minkowskian sublattice of Λ are the same that those we obtain by restricting ourselves to a *well rounded* lattice Λ, that is a lattice, the minimal vectors of which span E. For this reason, as in the title, we restrict ourselves to pairs (Λ, Λ') of a well-rounded lattice Λ and a sublattice Λ' generated by minimal vectors of Λ.

In this paper we consider thus the following problem: what is for a given dimension n the set of possible quotients Λ/Λ' for a given Λ as above when Λ' runs through the set of all sublattices of Λ having a basis made with minimal vectors of Λ?

Our results heavily rely on results obtained in [M1] (which extends previous work by Watson, Ryškov and Zahareva) in dimensions up to 8 and in [K-M-S] in dimension 9. One knows ([M1], theorem 1.7) that for Λ, Λ' as above, the index $[\Lambda : \Lambda']$ is bounded from above by $\gamma_n^{n/2}$ (γ_n is the *Hermite constant* for dimension n), an inequality which in particular bounds the annihilator d of Λ/Λ'. Then Λ is generated by a basis $\mathcal{B} = (e_1, \dots, e_n)$ of Λ' together with a finite set of vectors $e = \dfrac{a_1 e_1 + \dots a_n e_n}{d}$, defining this way a $\mathbb{Z}/d\mathbb{Z}$-code, namely the code with codewords $(a_1, \dots, a_n)$. In the two papers mentioned above, all the codes which may occur in a dimension $n \leq 9$ are listed (and in particular all possible quotients Λ/Λ'). But the existence of two given structures does not imply that they can be realized by sublattices

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of a same lattice Λ: for instance cyclic and non-cyclic quotients of order 4 exist in dimension 8, but whatever the dimension, no lattice Λ may have both these quotients without having sublattices with quotients cyclic of order 2.

The aim of this paper is to throw some light on the various existing combinations, according to the definition below, in which Λ' runs through the set of lattices having a basis made of minimal vectors of a given lattice Λ:

Definition 1.1. Let Λ be a well-rounded lattice.
1. The *maximal index* of Λ is $\imath(\Lambda) = \max_{\Lambda'} [\Lambda : \Lambda']$.
2. The *index system of* Λ, denoted by $\mathcal{I}(\Lambda)$, is the set of isomorphism classes of quotients Λ/Λ'.
3. We denote by $\mathcal{I}_n$ the union of index systems $\mathcal{I}(\Lambda)$ with $\dim \Lambda = n$.

When there is no risk of confusion, we shall write for short 4, $4 \cdot 2$, $4 \cdot 2^2$ to denote quotients isomorphic to $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$, respectively; and the notation (see Subsection 5.3)

$$\mathcal{I}(\Lambda_{75}) = \{1, 2, 3, 4, 2^2, 5, 6, 4 \cdot 2, 2^3\}$$

means that for the the lattice Λ_{75}, all structures up to order 8 except cyclic groups of order 7 or 8 may be realized by convenient sublattices having a basis of minimal vectors .

In Section 2 we recall some known results, mainly extracted from [M1] and [K-M-S]. Section 3 is devoted to dimension 6, index $\imath \le 3$ and related questions, Section 4 to dimension 7 and index 4, and Section 5 to dimension 8. Most of the constructions of lattices having a given index structure have been done using the *PARI-GP* package.

2. Minimal classes and codes

2.1. **General results.** As usual, $S(\Lambda)$ denotes the set of minimal vectors of the lattice Λ, and we set $s(\Lambda) = \frac{1}{2}\,|S(\Lambda)|$. Minimal classes are the equivalence classes of lattices for the relation

$$L \sim L' \iff \exists\, u \in \operatorname{End}(E) \mid u(L) = u(L') \text{ and } u(S(L)) = u(S(L'))\,,$$

equipped with the ordering defined by

$$\mathcal{C} \prec \mathcal{C}' \iff \exists \Lambda \in \mathcal{C},\, \exists \Lambda' \in \mathcal{C}' \mid S(\Lambda) \subset S(\Lambda')\,.$$

Clearly the index system of a lattice solely depends on its minimal class, and if $\mathcal{C} \prec \mathcal{C}'$, then the index structure of $\mathcal{C}$ is a subset of that of $\mathcal{C}'$. Also, to an n-dimensional class $\mathcal{C}$ we canonically attach its extension $\mathcal{C}_{ext}$ to dimension $n+1$, that of the lattices $\Lambda \perp \mathbb{Z}$ for $\Lambda \in \mathcal{C}$ scaled to minimum 1. Clearly $\mathcal{I}(\mathcal{C}_{ext}) = \mathcal{I}(\mathcal{C})$. However these trivial extensions will be useful to construct some "exotic" index systems; compare [M2] Section 3, or [M-S], Section 7.

Given n, d, and a code C over $\mathbb{Z}/d\mathbb{Z}$, among all minimal classes $\mathcal{C}$ on which C may be realized, if any, there exists a smallest one for the relation $\prec$, obtained using an averaging argument ([M1], Section 8; see also [M-S], Section 3 for a more general setting). Denote by $m \le n$ *the cardinality of* *the support of* C. The case of a binary code is easy: the smallest class is that of the lattices constructed by adjoining to $\Lambda' = \mathbb{Z}^n$ the vectors of the form $e = \frac{\sum a_i e_i}{2}$. In this case the e_i are pairwise orthogonal.

We now describe a notation we shall use for cyclic quotients. There is then a single vector $e = \frac{\sum a_i e_i}{d}$ to consider. We may assume that the a_i are zero for $m < i \le n$, and (negating some e_i if need be) that we have $1 \le a_1 \le \frac{d}{2}$ otherwise. We then denote by m_1 the number of subscripts j such that $a_j = i$ and by S_i' the set of vectors e_j for which $a_j = i$; we have $m_i \ge 0$ and $\sum_i m_i = m$. For a lifting of C to a pair (Λ, Λ') (if any), the scalar products may be chosen to have constant values x_i on S_i and $y_{i,j}$ on $S_i \times S_j$ (no $y_{i,j}$ if m_i or $m_j = 0$ and no x_i if $m_i \le 1$). We may moreover assume that $x_i = y_{i,j} = 0$ if d is even and $i = \frac{d}{2}$.

The complete description of codes is given in Table 11.1 of [M1] for $n \le 8$ and in various tables of [K-M-S] for $n = 9$, for instance in Table 2 for cyclic quotients, together with invariants relative to Λ and Λ': the kissing numbers s, s' and the *perfection rank* r *of* Λ (the set of similarity classes of lattices in the smallest class $\mathcal{C}$ depends on $\frac{n(n+1)}{2} - r$ parameters). Recall that a lattice (or a minimal class) with $r = \frac{n(n+1)}{2}$ is called *perfect*. Thus a perfect minimal class is the set of similarity classes of a perfect lattice.

2.2. **Calculation of index systems.** Once we know the index system $\mathcal{I}$ of such a class $\mathcal{C}$, we are sure that an index system which does not contain $\mathcal{I}$ cannot be realized using the corresponding code, which strongly limits the search of smaller index systems. As for the index systems of classes $\mathcal{C}' \succ \mathcal{C}$, they are contained in those of the perfect classes containing $\mathcal{C}$.

To calculate $\mathcal{I}(\Lambda)$ for a given lattice Λ, we can use the naive algorithm, consisting in extracting systems of n independent vectors of $S(\Lambda)$ and listing the corresponding quotients Λ/Λ'. Its complexity is roughly $\binom{s}{n}$. This is too large in case of the root lattice $\mathbb{E}_8$ ($\binom{s}{n} = \binom{120}{8}$), but works otherwise up to dimension 8, since by a theorem of Watson (Wat1), we have $s \le 75$ if we exclude $\mathbb{E}_8$. Dutour Sikirić has given in [K-M-S], Appendix B, a more efficient algorithm, which notably allowed him to deal with a lattice having $(s, n) = (99, 9)$.

2.3. **Specific lattices.** We list below a few results.

• Root lattices. These are the integral lattices which are generated by norm 2 vectors. They are orthogonal sums of the irreducible root lattices $\mathbb{A}_n$, $n \ge 1$, $\mathbb{D}_n$, $n \ge 4$, and $\mathbb{E}_n$, $n = 6, 7, 8$.

Proposition 2.1. *The index systems of irreducible root lattices are as follows:* $\mathcal{I}(\mathbb{A}_n) = \{1\}$; $\mathcal{I}(\mathbb{D}_n) = \{1, 2, \dots, 2^{\lfloor \frac{n-1}{2} \rfloor}\}$; $\mathcal{I}(\mathbb{E}_6) = \{1, 2, 3\}$; $\mathcal{I}(\mathbb{E}_7) = \{1, 2, 3, 4, 2^2, 2^3\}$; $\mathcal{I}(\mathbb{E}_8) = \{1, 2, 3, 4, 2^2, 5, 6, 4 \cdot 2, 2^3, 3^2, 2^4\}$.

Proof. The result for $\mathbb{A}_n$ is part of an old theorem of Korkine and Zolotareff; see [M2], Section 6.1. The other cases are dealt with using the classification of root systems; see [K-M-S], Appendix A, for $\mathbb{D}_n$ and [M1], Section 6 for $\mathbb{E}_n$. □

• <u>Perfect lattices</u>. The classification of perfect lattices is known up to dimension $n = 8$. Up to $n = 7$, disregarding root lattices and a few lattices with maximal index $\imath \le 2$, we are left with index systems $\{1,2,3\}$ ($n = 6,7$), and $\{1,2,3,4\}$ and $\{1,2,3,4,2^2\}$, this last one attained only on P_7^{10} (in Conway and Sloane's notation P_n^i; see [M2], Section 6.5).

In dimension 8, up to seven exceptions, the index systems share out among four types, namely

$\mathcal{I}_4 = \{1,2,3,4,2^2\},\ \mathcal{I}_5 = \mathcal{I}_4 \cup \{5\},\ \mathcal{I}_6 = \mathcal{I}_4 \cup \{6\},\ \text{and}\ \mathcal{I}_{5,6} = \mathcal{I}_4 \cup \{5,6\}\,.$

[The other systems, all attained on lattices having a perfect hyperplane section with the same minimum, are $\{1,2,3\}$ (twice), $\{1,2,2^2\}$, and those of the the three irreducible root lattices and of Barnes's lattice $\mathbb{A}_8^2 = \langle \mathbb{E}_7, \mathbb{A}_8 \rangle$, for which $\mathcal{I} = \mathcal{I}(\mathbb{E}_7)$.]

• <u>Maximal index systems</u>. The maximal index systems up to $n = 9$ are classified in [M1] and [K-M-S]. They reduce to $\{1\}$ if $n = 2,3$, attained on all lattices, and to $\{1,2\}$ if $n = 5$, attained on a 9-parameters family. For $n =$, 7, 8, they are attained uniquely on the perfect classes of $\mathbb{D}_4$, $\mathbb{E}_7$ and $\mathbb{E}_8$, respectively. For $n = 6$ and $n = 9$, there are two maximal systems. If $n = 6$, these are $\{1,2,2^2\}$, attained on $\mathbb{D}_6$, and $\{1,2,3\}$, attained on on a 10-parameters family. If $n = 9$, they are attained on the perfect classes of the laminated lattice Λ_9 and of a lattice denoted by L_{81} in [K-M-S]. We have

$$\mathcal{I}_9 = \{1,2,3,4,2^2,5,6,7,8,4\cdot 2,2^3,9,3^2,10,12,6\cdot 2,4^2,4\cdot 2^2,2^4\}\,,$$

$\mathcal{I}(\Lambda_9) = \mathcal{I}_9 \smallsetminus \{4^2\}$, and $\mathcal{I}(L_{81})$ consists of all structures of index up to 8 and the three structures of order 16.

• <u>Watson's identity</u>. We consider the case when Λ/Λ' is cyclic, writing $\Lambda = \langle \Lambda', e \rangle$ with $e = \dfrac{a_1 e_1 + \cdots + a_n e_n}{d}$. Denoting by $\operatorname{sgn}(x)$ the sign of the real number x, we have the identity

$$\Big(\big(\textstyle\sum_{i=1}^n |a_i|\big) - 2d\Big) N(e) = \textstyle\sum_{i=1}^n |a_i| \big(N(e - \operatorname{sgn}(a_i) e_i) - N(e_i)\big)\,,$$

which implies that when the a_i are strictly positive and add to $2d$, all the vectors $e - e_i$ are minimal. In this case, $\mathcal{I}(\Lambda)$ contains $\{1,2,\dots,d\}$.

3. Maximal index 3, dimension 6, and bases versus generators

In this section, we prove the classification results for lattices of maximal index $\imath \le 3$ or dimension $n \le 6$. We then consider some structures with $\imath = 4$ corresponding to lattices generated by their minimal vectors which do not have any basis of minimal vectors.

We shall give a common proof for the two theorems stated below. In the second one, we only list the new structures, those which do not exist in a lower dimension.

Theorem 3.1. *Let Λ be a lattice of maximal index $\imath \le 3$. The possible structure and the lower dimension $n_{\min}$ in which they exist are as follows:*

$\{1,2\}$: $n_{\min} = 4$; $\{2\}$: $n_{\min} = 5$; $\{1,2,3\}$: $n_{\min} = 6$; $\{3\},\{1,3\}$: $n_{\min} = 7$; $\{2,3\}$: $n_{\min} = 11$.

Theorem 3.2. *Let Λ be a lattice of dimension $n \le 6$. Then the minimal structures which exist in this dimension, but not in a lower one, are as follows:*
$n \le 3$: $\{1\}$; $n = 4$: $\{1,2\}$; $n = 5$: $\{2\}$; $n = 6$: $\{1,2,3\},\{1,2,2^2\}$.

Proof. By Watson's identity, $\imath = 2$ (resp. $\imath = 3$) is possible only if $n \ge 4$ (resp. $n \ge 6$), and if equality holds, the index structure must be $\{1,2\}$ (resp. $\{1,2,3\}$). That these lower bounds suffice can be seen in Table 11.1 of [M1]. This moreover shows that other systems of maximum index 2 (resp. 3) need $n \ge 5$ (resp. $n \ge 7$).

For any $n \ge 5$, taking $\Lambda = \Lambda' \cup (\frac{e_1+\cdots+e_n}{2} + \Lambda')$ with pairwise orthogonal vectors e_i, we obtain a lattice Λ with $S = S(\Lambda')$ (and $s = n$).

Let now $n = 7$ and $\Lambda = \Lambda' \cup \pm(\frac{e_1+\cdots+e_7}{2})$ with equal scalar products $e_i \cdot e_j = x_1$. Then for $x_1 = \frac{1}{5}$ (resp. $x_1 = \frac{1}{21}$), we obtain a lattice Λ with $S = S(\Lambda')$ and $s = n$ (resp. $S = S(\Lambda') \cup \{\pm e\}$ and $s = n+1$), and it is then evident that $\mathcal{I}(\Lambda) = \{3\}$ (resp. $\mathcal{I}(\Lambda) = \{1,3\}$).

Finally it proved in [M3], Lemma 3.2, that $\mathcal{I} = \{2,3\}$ needs $n \ge 11$, a lower bound which is optimal by a result of Conway and Sloane ([C-S]).

This completes the proof of Theorem 3.1.
(Nevertheless we shall sketch below a proof of the bound above and then adapt it to neighbour situations.)

By [M1], Table 11.1, if $n \le 3$, $n = 4$ or 5, $n = 6$, we have $\imath = 1$, $\imath = 2$, $\imath = 4$, respectively. As a consequence, the assertions of Theorem 3.2 result from the proof above of Theorem 3.1, except possibly for $n = 6$ and $\imath > 3$. But using again Table 11.1 of [M1], we see that this may occur only if $\imath = 4$ and $\Lambda \sim \mathbb{D}_6$, which implies that Λ/Λ' is 2-elementary. □

Proof of Theorem 3.1 for $1 \notin \mathcal{I}$. We write as above $\Lambda = \Lambda' \cup (\pm e + \Lambda')$ with $e = \frac{e_1+\cdots+e_m}{2}$ for some $m \le n$. Since $\mathcal{I}(\Lambda) \supsetneq \{1\}$, there is a minimal vector x in $e + \Lambda'$, say, $x = \frac{a_1 e_1+\cdots+a_n e_n}{3}$. By Watson's identity for denominator 3, we have $m \ge 6$, and even $m \ge 7$ since otherwise $e - e_1$ would be minimal. Since $\imath(\Lambda) = 3$, we have $|a_i| \le 3$, hence $a_i = 1$ or -2 if $i \le m$, and $a_i = 0, \pm 3$ if $i > m$. Again because $1 \notin \mathcal{I}$, we must have $a_i = -2$ if $i \le m$, and choosing n minimal, $a_i \ne 0$ if $i > m$. Watson's identity for denominator 2 now implies $1 + n - m \ge 4$, and even $1 + n - m \ge 5$ since otherwise $\frac{x+e_{m+1}+e_{m+2}+e_{m+3}}{2}$ would be minimal. Hence $n \ge m + (n-m) \ge 7 + 4 = 11$.
[Note that the averaging argument shows that three values suffice for the $e_i \cdot e_j$ (x_1 if $i < j \le m$, x_2 if $j > i > m$, y_1 if $i \le m < j$). The example of [C-S] is constructed this way.] □

The proof above applies directly to index systems $\{2,3,2^2\}$, and with slight modifications, to index systems containing $\{3,4\}$ but not $\{1,3,4\}$. However we may prove better results in some cases, but before analyzing more closely these lattices which are generated by their minimal vectors without having a basis of minimal vectors, we prove some lemmas.

Lemma 3.3. *Assume Λ has maximal index d and let $x = \frac{a_1 e_1 + \cdots + a_n e_n}{d} \in \Lambda$. If x is minimal, then $|a_i| \le d$ for all i.*

Proof. Let $\delta = \gcd(a_1, \dots, a_n)$, and set $a'_i = \frac{a_i}{\delta}$, $d' = \frac{d}{\delta}$, $\Lambda'' = \langle \Lambda', x\rangle$, and $\Lambda_0 = \langle x, e_j, j \ne i\rangle$. We have $[\Lambda'' : \Lambda'] = d'$, hence $[\Lambda : \Lambda''] = \frac{d}{d'}$.

Now let $i \in \{1, \dots, n\}$. If $a_i = 0$, there is nothing to prove. Otherwise we may write $e_i = \frac{\sum_{i \ne i} a_j e_j - d' x}{|a'_i|}$, with a denominator coprime with the gcd of the coefficients of the numerator. Hence we have $[\Lambda'' : \Lambda_0] = |a'_i|$, whence $[\Lambda : \Lambda_0] = \frac{d}{d'}\,|a'_i| \le d$, i.e., $|a'_i| \le d'$, and finally $|a_i| \le d$. □

Corollary 3.4. *Let Λ be a lattice of maximal index d, and suppose that we have $\Lambda = \langle \Lambda', f_1, \dots, f_\ell\rangle$ with $f_k = \frac{\sum_i a_i^{(k)} e_i}{d_k}$ and that $d = d_1 \cdots d_\ell$. If $x = \frac{\sum_i b_i e_i}{d_k} \in f_k + \Lambda'$ is minimal, then $|b_i| \le d_k$.*

Proof. Set $\Lambda'' = \langle \Lambda', f_k\rangle$. We have $[\Lambda'' : \Lambda'] = d_k$, hence $[\Lambda : \Lambda''] = \frac{d}{k}$ and $\imath(\Lambda'') \le \frac{\imath(\Lambda)}{d/d_k} = d_k$, whence the result by Lemma 3.3. □

Corollary 3.5. *Suppose that Λ/Λ' is 2-elementary of order 2^ℓ and that $\imath(\Lambda) = 2^\ell$. Then if $\mathcal{I}(\Lambda)$ strictly contains 2^ℓ, it contains $\{2^{\ell-1}, 2^\ell\}$.*

Proof. The hypothesis shows that $S(\Lambda)$ strictly contains $S(\Lambda')$, hence that there exists $x \in S(\Lambda) \smallsetminus \Lambda'$. Since Λ/Λ' is 2-elementary, x is of the form $\frac{\sum a_i e_i}{2}$. By Corollary 3.4, we have $|a_i| \le 2$, and since $x \notin \Lambda'$, a_i is odd for some subscript i. Let $\Lambda'' = \langle \Lambda', x\rangle$. Replacing e_i by x for such a subscript, we obtain a basis of minimal vectors for Λ'', and since Λ/Λ'' is 2-elementary, $2^{\ell-1}$ belongs to $\mathcal{I}(\Lambda)$. □

We now return to index systems for lattices of maximal index 4.

Proposition 3.6. (1) *If $\mathcal{I}(\Lambda) = \{2, 3, 4\}$ or $\{2, 3, 4, 2^2\}$, then $n \ge 11$.*
(2) *If $\mathcal{I}(\Lambda) = \{3, 4\}$, then $n \ge 15$.*

[By Corollary 3.5, the index system $\{3, 4, 2^2\}$ does not exist.]

Proof. We start as above with $\Lambda = \langle \Lambda', e\rangle$ and

$$e = \frac{e_1 + \cdots + e_{m_1} + 2(e_{m1+1} + \cdots + e_m)}{4} = \frac{e' + e_{m1+1} + \cdots + e_m}{2}$$

($m = m_1 + m_2 \le n$, $e' = \frac{e_1 + \cdots + e_{m_1}}{2}$). Since $3 \in \mathcal{I}(\Lambda)$, $S(\Lambda)$ is not contained in $S(\langle \Lambda', e'\rangle$, so that there exists a minimal vector $x \in e + \Lambda$, say, $x = \frac{a_1 e_1 + \cdots + a_n e_n}{4}$. As above we have $a_i = -3$ if $i \le m_1$, $a_i = \pm 2$ if $m_1 < i \le m$, $a_i = 0, \pm 4$ if $i > m$, and indeed $a_i \ne 0$ if n is minimal. Since $1 \notin \mathcal{I}$, e' cannot be minimal, which implies $m_1 \ge 5$. Using the denominator 3 provided by the a_i with $i \le m$, we see that we must have $1 + m_2 + (n - m) \ge 7$, i.e., $n \ge m_1 + 6 \ge 11$.

If moreover $2 \notin \mathcal{I}$, we must have $m_2 = 0$, hence $m_1 \geq 8$ by Watson's identity, and even $m_1 \geq 9$ because $e - e_1$ cannot be minimal, hence finally $n \geq m_1 + 6 \geq 15$. $\square$

The case of an index system $\{2, 3, 2^2\}$ is slightly more complicated.

Proposition 3.7. *If $\mathcal{I}(\Lambda) = \{2, 3, 2^2\}$ or $\{2, 3, 4, 2^2\}$, then $n \geq 13$.*

Proof. We denote by e, f, and $g \equiv e + f \mod \Lambda'$ representatives of the non-zero cosets of Λ/Λ', chosen so as to have components 0 or $\frac{1}{2}$. Since $1 \notin \mathcal{I}(\Lambda)$, the supports of e, f, g have cardinality at least 5 (i.e., the code has weight $w \geq 5$), which implies $m \geq 8$. We split $\{1, \dots, m\}$ into three sets I, J, K such that $e = \frac{\sum_{i \in I \cup J} e_i}{2}$ and $f = \frac{\sum_{i \in I \cup K} e_i}{2}$. Note that at least two of these sets are non-empty, and that exchanging e, f, we may assume that $|I| \geq 3$. Since the odd index 3 occurs in the index system, two of the cosets above, say, those of e and f, contain minimal vectors, say,

$$x = \frac{a_1 e_1 + \dots + a_n e_n}{2} \quad \text{and} \quad y = \frac{b_1 e_1 + \dots + b_n e_n}{2}\,.$$

By Corollary 3.4, we have $a_i = \pm 1$ on $I \cup J$, $b_i = \pm 1$ on $K \cup J$, and $a_i, b_i = 0, \pm 2$ otherwise, and not $(0, 0)$ (for $i > m$) if we choose n minimal.

Since $1 \notin \mathcal{C}$, the determinants $\left|\begin{smallmatrix} a_i & a_j \\ b_i & b_j \end{smallmatrix}\right|$ may not be equal to ± 1. This proves that $b_i, i \in I$ and $a_j, j \in K$ are non-zero: taking $i \in I$ and $j \in K$ if $K \neq \emptyset$ and $j \in J$ otherwise, we obtain the determinants $\left|\begin{smallmatrix} \pm 1 & a_k \\ b_i & \pm 1 \end{smallmatrix}\right|$ and $\left|\begin{smallmatrix} \pm 1 & \pm 1 \\ b_i & \pm 1 \end{smallmatrix}\right|$.

Now, negating if need be some e_i with $i \in I \cup J$, then y, then some e_i with $i \in K$, we may assume that $a_i = +1$ on $I \cup J$, $b_{i_0} = +2$ for some $i_0 \in I$, and $b_i = +1$ on K. On i_0 and $j \in J$, we have the determinant $\left|\begin{smallmatrix} 1 & 1 \\ b_0 & \pm 1 \end{smallmatrix}\right|$, hence $b_j = -1$. If $K \neq \emptyset$, since determinants ± 5 are excluded (because $\imath = 4$), we must have first $a_j = +2$ on K, then $b_i = +2$ on I. If $K = \emptyset$, we prove that $b_i = +2$ on I by using one index $j \in J$.

Now we have

$$2(x+y) = 3g + \sum_{i=m+1}^{n} (a_i + b_i) e_i\,, \ \text{ hence } \ x + y + \sum_{i=m+1}^{n} (a_i + b_i) e_i \equiv 0 \mod 3\,.$$

This shows that at least five terms $a_i + b_i$ must be non-zero, which implies $n \geq m + 5 \geq 13$ (and $a_i = b_i \pm 2$, $a_i = \pm 4$ and $b_i = \pm 4$ is impossible if $4 \notin \mathcal{I}$). $\square$

4. Maximal index 4 and dimension 7

The study of index 4 is organized as follows: we distinguish three types of index systems, those in which index 4 occurs with 4 alone, 2^2 alone, or both 4 and 2^2. In each case one has to consider 8 possible systems, corresponding to the eight subsets of $\{1, 2, 3\}$ (including $\emptyset$). We obtain this way 24 *a priori* possible systems. However, Corollary 3.5 shows that systems which strictly contain $\{2^2\}$ must contain $\{2, 2^2\}$, so that at least seven systems are impossible. We state this result as a proposition:

Proposition 4.1. *The seven index systems* $\{1,2^2\}$*,* $\{3,2^2\}$*,* $\{1,3,2^2\}$*,* $\{4,2^2\}$*,* $\{1,4,2^2\}$*,* $\{3,4,2^2\}$*,* $\{1,3,4,2^2\}$*. are impossible for a lattice of maximal index* 4. □

We know (Propositions 3.6 and 3.7) that the remaining four index systems containing 3 but not 1 (namely, $\{3,4\}$, $\{2,3,4\}$, $\{2,3,4,2^2\}$ and $\{2,3,2^2\}$) need $n \geq 11$, so that, in dimension 7, we are left with only 13 systems, among which $\{1,2,2^2\}$ exists in dimension 6. We shall prove that all other systems except possibly $\{1,2,3,2^2\}$ do exist, and give the corresponding minimal dimension.

Proposition 4.2. *The systems* $\{1,2,4,2^2\}$*,* $\{1,2,3,4\}$ *and* $\{1,2,3,4,2^2\}$ *exist in dimension* 7*, and for any other system* $\mathcal{I}$ *with* $\imath = 4$ *and* $4 \in \mathcal{I}$*, we must have* $(m_1, m_2) = (5,2)$ *or* $n \geq 8$.

Proof. We know by [M1] that $4 \in \mathcal{I}$ implies $n \geq 8$ or $(m_1, m_2) = (4,3), (5,2)$ or $(6,1)$. If $(m_1, m_2) = (4,3)$, the smallest class is obtained with pairwise orthogonal e_i, and we can check that we then have $\mathcal{I} = \{1,2,4,2^2\}$. Similarly, if $(m_1, m_2) = (6,1)$, Watson's identity shows that S contains the vectors $e - e_i$, hence that $\mathcal{I}$ contains $\{1,2,3,4\}$, and the averaging argument, taking $x_1 = \frac{1}{5}$ (the only possible choice) produces a lattice with $\mathcal{I} = \{1,2,3,4\}$. In both cases, the only larger system is $\{1,2,3,4,2^2\}$, that we know to exist (with $\Lambda = P_7^{10}$). □

Proposition 4.3. *It* $\mathcal{I} = \{1,4\}$ *or* $\{1,3,4\}$*, then* $n \geq 9$*, and these two systems exist in dimension* 9.

Proof. We keep the usual notation Λ, Λ', m_1, m_2, $m = m_1 + m_2$ and $e = \frac{\sum_{i=1}^n a_i e_i}{4}$. Since $1 \in \mathcal{I}$, the coset $e + \Lambda'$ contains a minimal vector x. If $m_2 > 0$, the numerator of x has a component ± 2, which implies $2 \in \mathcal{I}$. We thus have $m_2 = 0$, hence $m = m_1 \geq 8$, and if $m_1 = 8$, Watson's identity shows that the vectors $e - e_i, i \leq m$ are minimal, hence that $\mathcal{I}$ contains $\{1,2,3,4\}$. This proves the lower bounds $n \geq m = m_1 \geq 9$.

Take $m_2 = 0$ and consider the systems $S_1 = S(\Lambda') \cup \{\pm e\}$ and $S_2 = S(\Lambda') \cup \{\pm(e - e_1)\}$. It is easily checked that the index system of S_1 (resp. S_2) is $\{1,4\}$ (resp. $\{1,3,4\}$). Lattices with $n = m_1 = 9$ and $S = S_1$ or S_2 are constructed as follows. For S_1, take $e_i \cdot e_j = 7/72$ for $1 \leq i < j \leq 9$. For S_2, take $e_1 \cdot e_j = 9/40$ for $2 \leq j \leq 9$ and $e_i \cdot e_j = 7/40$ for $2 \leq i < j \leq 9$. □

Proposition 4.4. *If* $\mathcal{I} = \{2^2\}$ *or* $\{2,4,2^2\}$*, then* $n \geq 8$*, and these systems exist in dimension* 8.

Proof. Since $2^2 \in \mathcal{I}$, both systems may be constructed with a binary code of weight $w \geq 4$. Since a word of weight 4 lifts to a $\mathbb{D}_4$-section, the index system of a lattice constructed with a code of weight 4 contains 2. Hence, if $\mathcal{I} = \{2^2\}$, we must have $n \geq 8$, and this condition suffices, since there exists a (unique) binary code of length 8, dimension 2, and weight system $5^2 \cdot 6$.

Consider now the system $\{2,4,2^2\}$, and suppose that $n = 7$. Since $4 \in \mathcal{I}$, we can write Λ with Λ/Λ' cyclic of order 4, and since $\mathcal{I}$ contains 2 but not 1,

$S(\Lambda)$ spans its sublattice Λ'' which contains Λ' to index 2. Since $2^2 \in \mathcal{I}$, and since every code of length 7 and weight $w \ge 4$ has a word of weight 4, Λ'' has a $\mathbb{D}_4$-section. This shows that we may take $m_1 = 4$, hence $m_2 = 3$, but we know that the corresponding smallest class $\mathcal{C}$ has then index system $\{1, 2, 4, 2^2\}$, a contradiction.

This proves that $n \ge 8$, and taking $m_1 = m_2 = 4$ and pairwise orthogonal scalar products, we obtain a lattice with one quotient of type 2^2 and two of type (4).
[The averaging arguments applied on the one hand to cyclic quotients of order 4 with $m_1 = m_2 = 4$, and on the other hand to the binary code with weight system $(4\cdot 5\cdot 7)$ yield the same lattice, which accounts for the existence of quotients of both the types (4) and 2^2.] □

We are now able to give the complete list of index structure in dimension 7.

Theorem 4.5. *Let Λ be a lattice of dimension 7. Then the minimal structures which exist in this dimension are as follows:*

(1) $\imath \le 2$: $\{1\}$, $\{1, 2\}$, $\{2\}$.
(2) $\imath = 3$: $\{1, 2, 3\}$, $\{3\}$, $\{1, 3\}$.
(3) $\imath = 4$, $4 \in \mathcal{I}$, $2^2 \notin \mathcal{I}$: $\{4\}$, $\{2, 4\}$, $\{1, 2, 4\}$, $\{1, 2, 3, 4\}$.
(4) $\imath = 4$, $2^2 \in \mathcal{I}$, $4 \notin \mathcal{I}$: $\{2, 2^2\}$, $\{1, 2, 2^2\}$, $\{1, 2, 3, 2^2\}$.
(5) $\imath = 4$, $4 \in \mathcal{I}$, $2^2 \in \mathcal{I}$: $\{1, 2, 4, 2^2\}$, $\{1, 2, 3, 4, 2^2\}$.
(6) $\imath = 8$: $\{1, 2, 3, 4, 2^2, 2^3\}$, *attained uniquely on the class of* $\mathbb{E}_7$.

Proof. The case of $\imath > 4$ results from [M1], and that of $\imath \le 3$ results from Theorems 3.2 and 3.1. We are thus left with lattices of maximal index $\imath = 4$.
• $4 \in \mathcal{I}$ and $2^2 \notin \mathcal{I}$. Four out of the eight possible systems are excluded by Propositions 4.3 and 3.6, and $\{1, 2, 3, 4\}$ is known to exist by Proposition 4.2. We construct the three remaining structures using cyclic quotients with $(m_1, m_2) = (5, 2)$ and the unique parameter x_1 (by averaging, we may choose $x_2 = y_1 = 0$). With $x_1 = 3/20$, $x_1 = 1/4$, and any $x_1 \in (3/20, 1/4)$ (e.g., $x_1 = 1/5$), we obtain lattices with index systems $\{1, 2, 4\}$, $\{1, 4\}$, and $\{4\}$, respectively.
• $2^2 \in \mathcal{I}$, $4 \notin \mathcal{I}$. Five out of eight possible systems are excluded by Propositions 4.1, 4.4 and 3.7, so that we are left with the systems $\{1, 2, 2^2\}$ and $\{1, 2, 4, 2^2\}$, which are known to exist by Theorem 3.2 and Proposition 4.2, and the system $\{1, 2, 3, 2^2\}$. To construct an example having this index system, we observed that among perfect lattices, $4^2 \in \mathcal{I}$ holds only on $P_7^1 = \mathbb{E}_7$ and P_7^{10}, and then $\mathcal{I}$ contains both 4 and 2^2. This shows that minimal classes having the right system must lie below Voronoi paths connecting either of these two lattices. Among the eleven paths connecting two copies of $\mathbb{E}_7$, the one with $s = 32$ proved convenient. A computation with *PARI-GP* showed that index 1, 2, 3, 4 appears 923766, 21832, 90, and 6 times, respectively, the last case only with an elementary quotient. In all cases the binary code (of length 7) is the code with weight system $4 \cdot 5^2$. Here is a a Gram matrix belonging to this path (indeed, the eutactic one):

$$\begin{pmatrix} 4 & 2 & 2 & -2 & -2 & -1 & -1 \\ 2 & 4 & 2 & -2 & -2 & 1 & -2 \\ 2 & 2 & 4 & 0 & 0 & -1 & -2 \\ -2 & -2 & 0 & 4 & 2 & -1 & 0 \\ -2 & -2 & 0 & 2 & 4 & 0 & 0 \\ -1 & 1 & -1 & -1 & 0 & 4 & -1 \\ -1 & -2 & -2 & 0 & 0 & -1 & 4 \end{pmatrix}.$$

• $4 \in \mathcal{I}$, $2^2 \in \mathcal{I}$. Six out of eight systems are excluded by Propositions 4.1, 4.4 and 3.6, and the remaining two systems exist by Proposition 4.2. □

Theorem 4.6. *The seven index systems* $\{1, 2^2\}$, $\{3, 2^2\}$, $\{1, 3, 2^2\}$, $\{4, 2^2\}$, $\{1, 4, 2^2\}$, $\{3, 4, 2^2\}$ *and* $\{1, 3, 4, 2^2\}$ *do not exist. The other systems, except two for which existence is not known, are listed below together with the minimal dimension in which they exist:*

- $n_{\min} = 6$*:* $\{1, 2, 2^2\}$.
- $n_{\min} = 7$*:* $\{4\}$, $\{2, 4\}$, $\{1, 2, 4\}$, $\{1, 2, 3, 4\}$, $\{2, 2^2\}$, $\{1, 2, 3, 2^2\}$, $\{1, 2, 4, 2^2\}$, $\{1, 2, 3, 4, 2^2\}$.
- $n_{\min} = 8$*:* $\{2^2\}$, $\{2, 4, 2^2\}$.
- $n_{\min} = 9$*:* $\{1, 4\}$, $\{1, 3, 4\}$.
- $n_{\min} = 11$*:* $\{2, 3, 4\}$.
- $n_{\min} = 15$*:* $\{3, 4\}$.

For the systems $\{2, 3, 2^2\}$ *and* $\{2, 3, 4, 2^2\}$*, if any, we must have* $n_{\min} \geq 13$.

Proof. All the assertions above, are easy consequences of the results proved in this section and in the previous one, except those which concern $n_{\min} = 11$ or 15, for which we must construct lattices having convenient sets of minimal vectors.

An example for the index systems $\{2, 3, 4\}$, with $n = 11$, (resp. $\{3, 4\}$, with $n = 15$) has been obtained taking $(m_1, m_2) = (5, 6)$ (resp. $(9, 0)$), and using three values for the scalar products $e_i \cdot e_j$, x_1 for $i < j \leq m_1$, x_2 for $j > i > m_1$, and y_1 obtained as a function of x_1, x_2 for $i \leq m_1, j > m_1$. One may then take $(x_1, x_2) = (\frac{1}{9}, \frac{1}{4})$ (resp. $(\frac{1}{9}, \frac{1}{9})$).

We display below Gram matrices in the scale which make them integral and primitive, both with $s = n + 1$ minimal vectors, as in the proof of Proposition 3.6:

$$An11i234 = \begin{pmatrix} 8600 & 1756 & 1756 & 1756 & 1756 & 2135 & 2135 & 2135 & 2135 & 2135 & 2135 \\ 1756 & 1440 & 160 & 160 & 160 & 412 & 412 & 412 & 412 & 412 & 412 \\ 1756 & 160 & 1440 & 160 & 160 & 412 & 412 & 412 & 412 & 412 & 412 \\ 1756 & 160 & 160 & 1440 & 160 & 412 & 412 & 412 & 412 & 412 & 412 \\ 1756 & 160 & 160 & 160 & 1440 & 412 & 412 & 412 & 412 & 412 & 412 \\ 2135 & 412 & 412 & 412 & 412 & 1440 & 360 & 360 & 360 & 360 & 360 \\ 2135 & 412 & 412 & 412 & 412 & 360 & 1440 & 360 & 360 & 360 & 360 \\ 2135 & 412 & 412 & 412 & 412 & 360 & 360 & 1440 & 360 & 360 & 360 \\ 2135 & 412 & 412 & 412 & 412 & 360 & 360 & 360 & 1440 & 360 & 360 \\ 2135 & 412 & 412 & 412 & 412 & 360 & 360 & 360 & 360 & 1440 & 360 \\ 2135 & 412 & 412 & 412 & 412 & 360 & 360 & 360 & 360 & 360 & 1440 \end{pmatrix};$$

$$An15i34 = \begin{pmatrix} 1836 & 816 & 816 & 816 & 816 & 816 & 816 & 816 & 816 & 819 & 819 & 819 & 819 & 819 & 819 \\ 816 & 1728 & 192 & 192 & 192 & 192 & 192 & 192 & 192 & 364 & 364 & 364 & 364 & 364 & 364 \\ 816 & 192 & 1728 & 192 & 192 & 192 & 192 & 192 & 192 & 364 & 364 & 364 & 364 & 364 & 364 \\ 816 & 192 & 192 & 1728 & 192 & 192 & 192 & 192 & 192 & 364 & 364 & 364 & 364 & 364 & 364 \\ 816 & 192 & 192 & 192 & 1728 & 192 & 192 & 192 & 192 & 364 & 364 & 364 & 364 & 364 & 364 \\ 816 & 192 & 192 & 192 & 192 & 1728 & 192 & 192 & 192 & 364 & 364 & 364 & 364 & 364 & 364 \\ 816 & 192 & 192 & 192 & 192 & 192 & 1728 & 192 & 192 & 364 & 364 & 364 & 364 & 364 & 364 \\ 816 & 192 & 192 & 192 & 192 & 192 & 192 & 1728 & 192 & 364 & 364 & 364 & 364 & 364 & 364 \\ 816 & 192 & 192 & 192 & 192 & 192 & 192 & 192 & 1728 & 364 & 364 & 364 & 364 & 364 & 364 \\ 819 & 364 & 364 & 364 & 364 & 364 & 364 & 364 & 364 & 1728 & 144 & 144 & 144 & 144 & 144 \\ 819 & 364 & 364 & 364 & 364 & 364 & 364 & 364 & 364 & 144 & 1728 & 144 & 144 & 144 & 144 \\ 819 & 364 & 364 & 364 & 364 & 364 & 364 & 364 & 364 & 144 & 144 & 1728 & 144 & 144 & 144 \\ 819 & 364 & 364 & 364 & 364 & 364 & 364 & 364 & 364 & 144 & 144 & 144 & 1728 & 144 & 144 \\ 819 & 364 & 364 & 364 & 364 & 364 & 364 & 364 & 364 & 144 & 144 & 144 & 144 & 1728 & 144 \\ 819 & 364 & 364 & 364 & 364 & 364 & 364 & 364 & 364 & 144 & 144 & 144 & 144 & 144 & 1728 \end{pmatrix}.$$

□

5. Dimension 8

The list of structures with maximal index $\imath \leq 4$ can be extracted from Theorem 4.6. This list consists of the lattices listed in Theorem 4.5, together with the two systems $\{2^2\}$ and $\{2, 4, 2^2\}$.

For larger indices, the possible co-existence of 2^2 and 5 causes difficulties, as in the case of 2^2 and 3. For this reason, the existence of the structure $\{1, 2, 3, 2^2, 5\}$ remains open, whereas all other cases have been settled.

5.1. **Maximal index 5.** For maximal index 5, independently of the dimension, there are restrictions, as in Proposition 4.1, obtained with the same kind of proof: a system which contains $\{2^2, 5\}$ must contain $\{2, 2^2, 5\}$. There are also lower bounds better that $n \geq 8$ for some special systems, as in Proposition 4.3, related to the fact that one of the invariants m_1, m_2 must be equal to 2 or 3 if $n \leq 9$, which implies that systems $\{1, 5\}$ and $\{1, 4, 5\}$ do not exist if $n \leq 10$, and more precisely, that if $n \leq 10$, a system which strictly contains $\{5\}$ must contain $\{2, 5\}$ or $\{3, 5\}$. And we also know by [M-S] that if $\mathcal{I} \supset_{\neq} \{5\}$ and $1 \notin \mathcal{I}$, then $n \geq 10$; a 10-dimensional example, with index system $\{2, 3, 4, 5\}$, is given in [M-S].

In the general notation of [M1] for index 5, the cosets of Λ/Λ' are those of Λ', $\pm e + \Lambda'$ and $\pm e' + \Lambda'$, where

$$e = \tfrac{e_1 + \cdots + e_{m_1} + 2(e_{m_1+1} + \cdots + e_m)}{5} \quad \text{and} \quad e' = \tfrac{2(e_1 + \cdots + e_{m_1}) - (e_{m_1+1} + \cdots + e_m)}{5} \equiv 2e\,,$$

with $8 \leq m \leq n$ and $m_1 \geq m_2$. Here $n = m = 8$, and (m_1, m_2) must be equal to $(4, 4)$, $(5, 3)$ or $(6, 4)$. The smallest minimal class attached to a pair (m_1, m_2) is invariant under the action of $S_{m_1} \times S_{m_2}$ and can be constructed using three parameters x_1, x_2, y_1, namely the scalar products $e_i \cdot e_j$ for $i < j \leq m_1$, for $m_1 < i < j$ and for $i \leq m_1, j > m_1$, respectively. The corresponding sets of minimal vectors (which have $s = 16$, $s = 8$, $s = 16$) together with possible choices for the parameters (e.g., $(\frac{1}{4}, \frac{1}{4}, 0)$, $(\frac{1}{4}, \frac{1}{8}, \frac{1}{16})$, $(\frac{3}{10}, \frac{1}{8}, \frac{1}{8})$) are given in [M1], and we easily deduce from these data that the index systems are $\{1, 2, 3, 5\}$, $\{5\}$, and $\{1, 2, 3, 4, 5\}$, respectively.

Theorem 5.1. *The index system of an* 8*-dimensional lattice of maximal index* 5 *is one of* $\mathcal{I}_1 = \{1, 2, 3, 4, 2^2, 5\}$*,* $\mathcal{I}_2 = \{1, 2, 3, 4, 5\}$*,* $\mathcal{I}_3 = \{1, 2, 3, 5\}$*,* $\mathcal{I}_4 = \{1, 2, 4, 5\}$*,* $\mathcal{I}_5 = \{1, 2, 5\}$*,* $\mathcal{I}_6 = \{5\}$*, and maybe* $\{1, 2, 3, 2^2, 5\}$*.*

Proof. The proof will involve three steps: (1) the construction of more index systems; (2) the proof that an index system which strictly contains $\{5\}$ indeed contains $\{1, 2, 5\}$; (3) the proof that an index system $\mathcal{I}$ with $3 \notin \mathcal{I}$ must be equal to $\{1, 2, 5\}$ or to $\{1, 2, 4, 5\}$.

(1) Choose $(m_1, m_2) = (5, 3)$. Taking $(x_1, x_2, y_1) = (\frac{1}{4}, \frac{1}{12}, \frac{1}{60})$, we obtain a lattice with $S = \{\pm e_i, \pm e\}$, hence $\mathcal{I} = \{1, 2, 5\}$; taking $(x_1, x_2, y_1) = (\frac{1}{4}, -\frac{1}{12}, \frac{1}{12})$, we obtain a lattice with $s = 30$ and $\mathcal{I} = \{1, 2, 3, 4, 2^2, 5\}$. [This last index system (but no smaller system) occurs for numerous perfect lattices.]

There remains to construct a lattice with $\mathcal{I} = \{1, 2, 4, 5\}$. To this end we now use 5 parameters, restricting x_1 to $i, j \le 4$ and y_1 to $i \le 4$, introducing $z_1 = e_i \cdot e_5$ $(i \le 4)$, $z_2 = e_5 \cdot e_j$ $(j \ge 6)$, then setting $z_2 = \frac{1}{4}x_1 + \frac{1}{2}x_2 + y_1 - \frac{2}{3}z_1 + \frac{7}{48}$ to ensure $e - e_5 \in S$. Taking $(x_1, z_1, x_2, y_1) = (\frac{1}{5}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16})$, we obtain a lattice with $S = \{\pm e_i, \pm(e - e_5)\}$, hence $\mathcal{I} = \{1, 2, 4, 5\}$.

(2) Let Λ be a lattice with $\mathcal{I} \supsetneq \{5\}$. There is nothing to prove if $(m_1, m_2) = (4, 4)$ or $(6, 2)$, and we may assume that $S \supsetneq S(\Lambda')$, hence that there exists a minimal vector $x \in e + \Lambda'$ or $x' \in e' + \Lambda'$. Then the coefficients a_i (resp. a'_i) in the numerator of x (resp. of x') are 1 or -4 if $i \le 5$ and 2 or -3 if $i \ge 6$ (resp. 2 or -3 if $i \le 5$ and -1 or $+4$ if $i \ge 6$). The coefficients a'_i, $i \le 5$ of an $x' \in S(e' + \Lambda')$ cannot all be equal to -3, since an index 3 would then exist in dimension 4. Thus $a'_i = 2$ for some i, which proves (2) if $S(e' + \Lambda') \ne \emptyset$.

If there exists $x \in S(e + \Lambda')$ with all $a_1 = -3$, Watson's identity with denominator 3 shows that Λ contains a vector $y = \frac{x + b_1 e_1 + \cdots + b)5 e_5}{3}$ with $b_i = \pm 1$ ($b_i \equiv a_i\,(= 1\,\text{or}\,-4)$ mod 3. Then $y - b_1 e_1$ is a minimal vector in $e' + \Lambda'$, a contradiction.

(3) The proof will be a consequence of the following lemma:

Lemma 5.2. *If* $3 \notin \mathcal{I}(\Lambda)$, *then* $S(\Lambda) \subset T = \{\pm e_1, \dots, \pm e_8 \pm e, \pm e', \pm(e - e_i)\}$ *for* one *index* $i \le 5$.

Indeed, it is readily verified that $\mathcal{I}(T) = \{1, 2, 4, 5\}$, which excludes the structures $\{1, 2, 2^2, 5\}$ and $\{1, 2, 2^2, 4, 5\}$.

This completes the proof of Theorem 5.1. □

Proof of Lemma 5.2. We successively consider the cosets of 0, e and e' in Λ modulo Λ', using the notation e, e', a_i, a'_i introduced in the proof of (2) above.

(1) Because of the bound $\imath(\Lambda) \le 5$, the components of the minimal vectors of Λ' on the e_i must be 0 or ± 1. We must discard vectors of the form $e_1 + e_2$ or $e_6 + e_7$, since we could the write e using 7 independent vectors in its numerator; and a base change will show that using a vector of the form $e_1 - e_2$, $e_6 - e_7$ or $e_1 \pm e_6$, we may define Λ with $(m_1, m_2) = (4, 4)$ or $(6, 2)$. Using this remark, we easily see that if there were in $S(\Lambda')$ a sum $e_i \pm e_j \pm e_k$ with more than two components, then we could again express e using less than 8 vectors in its numerator.

(2) the minimal vectors $x \in e + \Lambda'$ must have $a_i = 1$ or -4 if $i \le 5$ and $a_i = 2$ if $i > 5$, and not three or more a_i equal to -4, since otherwise we would have an index 4 in a dimension smaller than 7.

If, say, $a_1 = a_2 = -4$, we have

$$e = x + e_1 + e_2 = \frac{(-x + e_3 + e_4 + e_5)/2 + e_6 + e_7 + e_8}{2},$$

which shows that $\frac{\pm x \pm e_3 \pm e_4 \pm e_5}{2}$ are minimal. Setting $y = \frac{x + e_3 + e_4 + e_5}{2}$, we then have

$$y = \frac{e_6 + e_7 + e_8 - 2(e_1 + e_2) + 3(e_3 + e_4 + e_5}{5},$$

and an index 3 shows up.

Finally, if, say, $e - e_1$ and $e - e_2$ are minimal, the identity

$$e = \frac{-(e-e_1)-(e-e_2)+e_3+e_4+e_5+2(e_6+e_7+e_8)}{5}$$

with 8 independent vectors in the numerator shows the existence of an index 3. This proves that $S(e+\Lambda')$ must be a subset of $\{\pm e, \pm(e-e_i)\}$ for *one* $i \in \{1,2,3,4,5\}$.

(3) We must have $a'_i = 2$ for $i \le 5$, and if some a'_i were equal to 4 for $i \ge 6$, then we would an index 2 in a dimension least than 4. This proves that $S(e'+\Lambda')$ must be a subset of $\{\pm e'\}$. $\square$

Here is a Gram matrix ($n = 8$, $\imath = 5$, $s = 9$) with $S = \{\pm e_i, \pm(e-e_5)\}$:

$$\begin{pmatrix} 1404 & 534 & 534 & 534 & 702 & 697 & 697 & 697 \\ 534 & 1200 & 240 & 240 & 300 & 75 & 75 & 75 \\ 534 & 240 & 1200 & 240 & 300 & 75 & 75 & 75 \\ 534 & 240 & 240 & 1200 & 300 & 75 & 75 & 75 \\ 702 & 300 & 300 & 300 & 1200 & 185 & 185 & 185 \\ 697 & 75 & 75 & 75 & 185 & 1200 & 150 & 150 \\ 697 & 75 & 75 & 75 & 185 & 150 & 1200 & 150 \\ 697 & 75 & 75 & 75 & 185 & 150 & 150 & 1200 \end{pmatrix}.$$

5.2. **Maximal index 6.** Listing the various combinations of maximal index 6, with or without 2^2 and / or 5, looks very complicated beyond $n = 8$, though the codes are known in all dimensions ([K-M-S], Section 6). Thus we restrict ourselves to dimension $n = 8$.

Theorem 5.3. *The index system of an* 8*-dimensional lattice of maximal index* 6 *is one of the three systems*

$\mathcal{I}_1 = \{1,2,3,4,2^2,5,6\}$, $\mathcal{I}_2 = \{1,2,3,4,2^2,6\}$ *or* $\mathcal{I}_3 = \{2,4,2^2,6\}$.

Proof. In [M1], table 11.1, six types of maximal index 6 are listed, among which we must discard the third one, which only exists for the class of $\mathbb{E}_8$. Using the data of this table, we can determine the index system of the smallest minimal class in each case. Here are the results for each remaining five sets (m_1, m_2, m_3): $(4,3,1), (3,4,1): \mathcal{I}_1$; $(2,4,2), (4,2,2): \mathcal{I}_2$; $(3,3,2)$: $\mathcal{I}_3$.

This shows first that the three structures listed above exist, and next that a further structure, if any, must strictly contain $\mathcal{I}_3$ and must be realized using $(m_1, m_2, m_3) = (3,3,2)$. To deal with this case, we introduce the notation

$$e = \frac{e_1+e_2+e_3+2(e_4+e_5+e_6)+3(e_7+e_8)}{6},\ e' = \frac{e_1+e_2+e_3-e_4-e_5-e_6}{3},\ e'' = \frac{e_1+e_2+e_3+e_7+e_8}{2}.$$

By Watson's identity, the 6 vectors $e' - e_i$, $e' + e_j$, $i = 1,2,3$, $j = 4,5,6$ are minimal. For a sublattice L of E with $S(L) \subset \Lambda' \cup (e'+\Lambda')$, we have $[\Lambda : L] = 2$, hence $\mathcal{I}(L) \subset \mathcal{I}_3$. Hence a lattice L with $\mathcal{I}(L) \supsetneq \mathcal{I}_3$ must have a minimal vector x off the cosets of 0 and e', and moreover $\mathcal{I}(L)$ must contain and odd number. Then its minimal vectors generate L, so that by [M3], we have $1 \in \mathcal{I}(L)$. To prove the theorem, it suffices to show that $\mathcal{I}(L)$ then also contains 3. This we now prove.

If $\pm x \in e + \Lambda'$, let $x = \frac{a_1 e_1 + \cdots + a_8 e_8}{6}$. For $i = 7$ or 8, we have $\pm a_i \equiv 3 \bmod 6$, hence $a_i = \pm 3$, and the existence of an index 3 is clear.

Let now $x = \frac{a_1e_1+\cdots+a_8e_8}{2} \in e'' + \Lambda'$. The a_i are odd for $i = 1,2,3,7,8$ and even for $i = 4,5,6$. We first show that $a_i = \pm 1$ for $i = 1,2,3$. We have $e_1 = 2x - a_2e_2 - \cdots - a_8e_8$, so that e may be written on the independent vectors $x, e_2, \dots, e_8$ in the form $e = \pm\dfrac{2x + b_2e_2 + \cdots + b_8e_8}{6|a_1|}$. Since the gcd of the coefficients of the numerator is 1 or 2, $3|a_1|$ is an index for L, which implies $3|a_1| \le 6$, hence $a_1 = \pm 1$, and similarly $a_2, a_3 = \pm 1$.

Permuting e_1, e_2, e_3 and negating x if need be, we may assume that $a_1 = a_2 = +1$ and write $\pm e = \pm\dfrac{2x + b_3e_3 + \cdots + b_8e_8}{6}$ with $b_3 = 0$ or -1 as a combination of seven minimal vectors with denominator 6. Since index 6 is not possible in dimension 7, all b_i must be even, and in particular, b_3 must be zero. Now x is a combination of six minimal vectors with denominator 3 and coprime coefficients in the numerator. Watson's identity for denominator 3 shows that $e + \Lambda'$ contains minimal vectors, and we are back to the first case. □

5.3. **Maximal index 8.** We know from [M1] that we have $\imath \le 8$ except on the class of $\mathbb{E}_8$ (see Section1), where there exists elementary quotients Λ/Λ' of order 9 and 16, and that cyclic quotients of order 7 or 8 do not exist in dimension 8. Six codes for index 8 are listed in Table 11.1 of [M1], $n = 8$. We denote the corresponding smallest minimal classes by $\mathcal{C}_{8a}$ to $\mathcal{C}_{8f}$, and by $\mathcal{C}8g$ that of $\mathbb{E}_7 \oplus \mathbb{A}_1$, which extends $\mathrm{cl}(\mathbb{E}_7)$ to $n = 8$; the quotient Λ/Λ' is of type $(4 \cdot 2)$ in the first three cases, and 2-elementary in the remaining four cases. The class $\mathcal{C}_{8f}$ (with $(s,r) = (32,23)$) is that of the lattice L_{32} which lifts the unique binary code having weight system $(4^3 \cdot 5^4)$. The class $\mathcal{C}_{8b}$ is a Voronoi path $\mathbb{E}_8$ — $\mathbb{E}_8$ (with $(s,r) = (75,35)$) discovered by Watson, along which lattices have an $\mathbb{E}_7$-section (and also a $\mathbb{D}_7$-section). The first three codes define quotients of type $4 \cdot 2$, the remaining four elementary quotients. Averaging on codes for classes $\mathcal{C}_{8a}$ and $\mathcal{C}_{8e}$ yields isometric lattices, with $(s,r) = (48,32)$, hence $\mathcal{C}_{8a} = \mathcal{C}_{8e}$.

We display below Gram matrices M_{32} for L_{32} and W_{75} for the eutactic lattice Λ_{75} lying on the Watson path; the basis for L_{32} is $(e_1, e_2, e_3, e, e_5, f, e_7, g)$ where

$$e = \frac{e_1+e_2+e_3+e_4}{2},\ f = \frac{e_3+e_4+e_5+e_6}{2} \ \text{ and } \ g = \frac{e_2+e_4+e_6+e_7+e_8}{2},$$

and $(e_1, \dots, e_8)$ is an orthogonal basis for Λ':

$$M32 = \begin{pmatrix} 4&0&0&2&0&0&0&0\\ 0&4&0&2&0&0&0&2\\ 0&0&4&2&0&2&0&0\\ 2&2&2&4&0&2&0&2\\ 0&0&0&0&4&2&0&0\\ 0&0&2&2&2&4&0&2\\ 0&0&0&0&0&0&4&2\\ 0&2&0&2&0&2&2&5 \end{pmatrix} \ ; \quad W75 = \begin{pmatrix} 4&2&2&2&2&2&2&1\\ 2&4&0&0&0&2&0&2\\ 2&0&4&2&2&0&0&0\\ 2&0&2&4&2&0&0&0\\ 2&0&2&2&4&0&0&0\\ 2&2&0&0&0&4&0&0\\ 2&0&0&0&0&0&4&0\\ 1&2&0&0&0&0&0&4 \end{pmatrix}.$$

We shall prove the following result:

Theorem 5.4. *The index system of an* 8*-dimensional lattice* Λ *with* $\imath(\Lambda) > 6$ *is one of the following five systems:*

$\mathcal{I}_1 = \{1,2,3,4,2^2,5,6,4\cdot 2,2^3,3^2,2^4\}$, $\mathcal{I}_2 = \{1,2,3,4,2^2,5,6,4\cdot 2,2^3\}$,

$$\mathcal{I}_3 = \{1,2,3,4,2^2,4.2,2^3\},\ \mathcal{I}_4 = \{1,2,3,4,2^2,2^3\}\,,$$
$$\mathcal{I}_5 = \{1,2,2^2,2^3\},\ \mathcal{I}_6 = \{2,4,2^2,2^3\}\,.$$

All these systems exist, $\mathcal{I}_1$, $\mathcal{I}_5$, $\mathcal{I}_6$ on unique minimal classes, that of $\mathbb{E}_8$, $\mathbb{D}_8$ and L_{32}, respectively, and $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$, on several classes. The system $\mathcal{I}_2$ is that of the Watson path, $\mathcal{I}_3$ that of $\mathcal{C}_{8a} = \mathcal{C}_{8e}e$, and $\mathcal{I}_5$ that of $\mathbb{E}_7 \perp \mathbb{A}_1$ and also of one well-defined class $\mathcal{C}'_{8f} \succ \mathcal{C}_{8f}$ with $(s,r) = (33,24)$.

Proof. We first list the invariants (s,r) and $\mathcal{I}$ of the six smallest minimal classes related to the seven codes listed above:
$\mathcal{C}_{8a}$: $(s,r) = (48,32)$, $\mathcal{I} = \mathcal{I}_3$ ($\mathcal{C}_{8e} = \mathcal{C}_{8a}$);
$\mathcal{C}_{8b}$: $(s,r) = (75,35)$, $\mathcal{I} = \mathcal{I}_2$;
$\mathcal{C}_{8c}$: $(s,r) = (120,36)$, $\mathcal{I} = \mathcal{I}_1$ ($\mathcal{C}_{8c} = \mathrm{cl}(\mathbb{E}_8)$);
$\mathcal{C}_{8d}$: $(s,r) = (56,36)$, $\mathcal{I} = \mathcal{I}_5$ ($\mathcal{C}_{8d} = \mathrm{cl}(\mathbb{D}_8)$);
$\mathcal{C}_{8f}$: $(s,r) = (32,23)$, $\mathcal{I} = \mathcal{I}_6$;
$\mathcal{C}_{8g}$: $(s,r) = (64,29)$, $\mathcal{I} = \mathcal{I}_4$.

Here are a few comments on the list above. From [M1], we know $\mathcal{I}_1$ and $\mathcal{I}_4$, and the fact that we have $\imath \le 8$ except on the class of $\mathbb{E}_8$. This shows that every index system except $\mathcal{I}_1$ is contained in $\mathcal{I}_2$.

We also know that $\mathcal{C}_{8b}$ contains $\mathcal{I}(\mathbb{E}_7)$ and $\{4 \cdot 2\}$. A computer search then quickly finds cyclic quotients of order 5 and 6 (a few days computations finds the number of occurrences of all quotients Λ/Λ'), and the remaining calculations are much shorter. This proves the existence of the six index systems of Theorem 5.4. Note also that that 4 belongs to all systems except $\mathcal{I}_5$. We have thus also proved the uniqueness assertions about $\mathcal{I}_1$ and $\mathcal{I}_5$.

To classify all index systems containing $4 \cdot 2$, it now suffices to consider classes containing $\mathcal{C}_{8a}$. The perfection co-rank of $\mathcal{C}_{8a}$ is sufficiently small ($36 - 32 = 4$) to allows us to find all classes $\mathcal{C}$ lying above $\mathcal{C}_{8a}$ (in other words, to find its *Ryshkov polyhedron* in the sense of [K-M-S], Section 3). One class has $(s,r) = (49,33)$ and $\mathcal{I} = \mathcal{I}_3$. the other classes all have $\mathcal{I} = \mathcal{I}_2$ except the maximal one which is that of $\mathbb{E}_8$. (These have invariants $(33,56)$, $(34,57)$, $(35,66)$, and $(35,75)$, the Watson path.)

This proves that the index systems containing $4 \cdot 2$ are $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$.

We now turn classes which lie above $\mathcal{C}_{8g}$ and have an index system which strictly contains $\mathcal{I}(\mathcal{C}_{8g}) = \mathcal{I}_4$. The maximal classes are those of a perfect lattice with $\imath \ge 8$. The only such lattice is $\mathbb{E}_8$. By the results of [D-S-V], classes with $r \le 35$ are contained in the Watson path. While classifying the possible values of s in dimension 8, the authors of [D-S-V] have proved that there exists a unique class with $r = 34$ lying below $\mathcal{C}_{75} = \mathcal{C}_{8b}$, which has $s = 69$. I have checked that this class $\mathcal{C}_{69}$ has again index system $\mathcal{I}_2$.

Now we have $s - r = 35$ on $\mathcal{C}_{69}$ as on $\mathcal{C}_{8g}$. This shows that the classes $\mathcal{C}$ such that $\mathcal{C}_{8g} \prec \mathcal{C} \prec \mathcal{C}_{69}$ are obtained by removing arbitrary vectors off $\mathbb{E}_7$ from $\mathcal{C}_{69}$. Testing equivalence, we have shown that there are two such classes with $r = 33$, and have checked that all have $\mathcal{I} = \mathcal{I}_4$.

This proves that the classes containing $\mathcal{C}8g$ have index system $\mathcal{I}_1, \mathcal{I}_2$ or $\mathcal{I}_4$.

Finally we are left with classes $\mathcal{C} \succ \mathcal{C}_{8f}$. Classifying all possible classes is certainly complicated, since the minimal class $\mathcal{C}_{8f}$ depends on $36 - 23 = 13$ parameters, namely the scalar products $e_i \cdot e_j$ for $i = 1, \dots, 6$ and $j = 7, 8$, and $e_7 \cdot e_8$. Thanks to Lemma 5.5 below, we can avoid such a classification. The matrix M_{32} is obtained taking these parameters all zero. Replacing 0 by $-\frac{1}{12}$ for $i = 2, 4, 6$ and $j = 7, 8$, we obtain a lattice with $s = 33$, $r = 24$ and $\mathcal{I} = \mathcal{I}4$. Its minimal class is the class $\mathcal{C}'_{8f}$.

Lemma 5.5. *Let $\mathcal{C}$ be a minimal class containing* strictly $\mathcal{C}_{8f}$. *Then one of the following assertions holds:*

(1) $\mathcal{C} \succ \mathcal{C}_{8g}$.
(2) $\mathcal{I}(\mathcal{C}) \supset \mathcal{I}_2$.
(3) $\mathcal{C} \succ \mathcal{C}'_{8f}$, *and $\mathcal{C}$ can be defined by a set of minimal vectors contained in $S(\mathcal{C}_{8f}) \cup \{\frac{\pm e_2 \pm e_4 \pm e_6 \pm e_7 \pm e_8}{2}\}$.*

In all cases, $\mathcal{I}(\mathcal{C})$ contains $\mathcal{I}_4$.

Taking for granted this lemma, we can now complete the proof of Theorem 5.4. The last assertion of the lemma proves that only $\mathcal{C}_{8f}$ has index system $\mathcal{I}_6$. Next a computer calculation on the few systems of minimal vectors as in (3) shows that either $\mathcal{I}(\mathcal{C}) = \mathcal{I}_4$ or $\mathcal{I}(\mathcal{C})$ contains $\mathcal{I}_2$. □

Proof of Lemma 5.5. Let $\mathcal{C} \succ_{\neq} \mathcal{C}_{8g}$, let $\Lambda \in \mathcal{C}$, and let $x \in S(\Lambda) \smallsetminus S(\Lambda_{32})$, belonging to a coset $v + \Lambda'$. We consider three cases:
(1) $v = 0$ (i.e., $x \in \Lambda'$);
(2) v lifts a word of weight 4 (i,e., $v = e, f$ or $e + f$);
(3) v lifts a word of weight 5 (i,e., $v = g, g + e, g + f$ or $g + e + f$).
Taking into account the automorphisms of the code, we may assume that $v = e$ in case (2) and $v = g$ in case (3). In all cases, by Corollary 3.4, the components of x on the basis $(e_1, \dots, g)$ used to construct Λ_{32} are $0, \pm 1$.

(1) Let $x = \pm e_{i_1} \pm \cdots \pm e_{i_k}$ (with one or two terms in $\{e_7, e_8\}$, since $e_i \cdot e_j = 0$ if $i < j < 7$). If $k = 2$, replacing e_7 or e_8 by x amounts to change the code into a code of length 7 generated by weight-4 words, that is, the code of $\mathbb{E}_7$. There cannot be three components in the support of a weight-5 word, and if, say, $x = e_1 + e_2 + e_7$, then $e_1 + e_7$ is minimal since $e_1 \cdot e_2 = 0$. The case when $k \geq 4$ similarly reduces to $k - 1$.

(2) We may assume using change of signs that $x = e \pm e_7$ or $e \pm e_7 \pm e_8$. Replacing e_1, e_2, e_3, e_4 by the four vectors $\frac{e_1 \pm e_2 \pm e_3 \pm e_4}{2}$ having 0 or 2 minus signs, we are back to the previous case.

(3) We have $v = g$, so that the minimal class of $\langle \Lambda, x \rangle$ is either $\mathcal{C}'_{8g}$, or x may be assumed to be equal to $g + e_1$, $g + e_1 + e_3$, or $g + e_1 + e_3 + e_5$. The last two cases reduce to $x = g + e_1$ (because $e_1 \cdot e_3 = 0$), and a computer calculation shows that $\mathcal{I}$ then contains $\mathcal{I}_2$.

Moreover, if $\mathcal{C}$ contains besides g a vector $y \neq \pm g$, then either y belongs to $g + \Lambda'$, and then y is a vector g' obtained from g by changing signs of some

e_i, or y belongs to $e+g$, say, and using the argument used deal with (2), we again reduce ourselves to the previous situation. □

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