# Réseaux et designs sphériques

**Jacques Martinet**
**(Laboratoire A2X, Uni. Bordeaux 1)**

**Besançon, le 16 octobre 2003**

Colloque en l'honneur de
**Georges** GRAS

## Combinatorics and lattices

A few years ago, Boris Venkov discovered that there are some interesting connections between the relatively recent theory of **spherical designs** on the one hand, and the theory of **extreme lattices** initiated by Alexandre Korkine and Igor Zolotareff in their 1877 paper and developed thirty years later by Georges Voronoï.

A *lattice* is a discrete subgroup of a Euclidean space $E$, of maximal rank, indeed $n = \dim E$. A lattice $\Lambda$ is *extreme* if the density of the sphere packing canonically attached to any lattice attains a local maximum at $\Lambda$.

Our spherical designs will live on the sphere of minimal vectors of a lattice; more generally, we shall sometimes consider the various layers; minimal vectors in the **dual lattice** will often play an important rôle.

## References

Basic book: Réseaux euclidiens, designs sphériques et groupes, L'Enseignement Mathématique, Monographie **37**, J. Martinet, ed., Genève, 2001; see in particular the contributions of Venkov, Bachoc–Venkov, Martinet, Martinet–Venkov.

Further related papers:

Gabi Nebe–Venkov, *The strongly perfect lattices in dimension* 10, J. TdN Bx, **12** (2000), 503–518.

A.-M. Bergé–Martinet, *Symmetric Groups and Lattices*, Monatshefte Math. (2003), to appear.

Martinet–Venkov, *On integral lattices having an odd minimum*, preprint, 42. pp.

## The notion of a spherical design

Let $S^{n-1}$ be the unit sphere with center $O$, endowed with the standard measure scaled to volume $1$, let $t > 0$ be an integer, and let $X \subset \Sigma$ be a finite set.

We say that $X$ is a *(spherical)* **t**-*design* if

$$\int_{\Sigma} f\,dx = \frac{1}{|X|} \sum_{x \in X} f(x)$$

holds for all polynomials of degree at most $\mathbf{t}$ on $E$.

Equivalent definition: the integral above is zero for all *homogeneous, harmonic polynomials* of degree at most $\mathbf{t}$.

**Example 1** *"$X$ is a $1$-design" $\iff$ "$0$ is the center of gravity of $X$".*

**Remark 1** Any symmetric set which is a $2t$-design is a $(2t+1)$-design.

**Remark 2** If $n = 1$, every $2$-design is a $t$-design for all $t$.

## Design identities

**From now on, all designs are symmetric.**

**Theorem 1** *If* $n \geq 2$ *and if* $t \geq 2$ *is* even, *the following conditions are equivalent:*

1. $X$ *is a* $t$*-design.*
2. *For all even* $p \leq t$*, there exists a constant* $c_p$ *such that for all* $\alpha \in E$*,*

$$\sum_{x \in X} (x \cdot \alpha)^p = c_p (\alpha \cdot \alpha)^{p/2} (x \cdot x)^{p/2}.$$

3. *The identity above holds for* $p = t$*.*

*Moreover, when these conditions hold, we have*

$$c_p = \tfrac{1.3.5 \ldots (p-1)}{n(n+2) \ldots (n+p-2)} \left| X \right|.$$

[However, to consider all even integers $p \leq t$ may prove useful.]

## Some notation for lattices

The *norm of* $x \in E$ is $N(x) = x \cdot x$.
The *minimum of* $\Lambda$ is $m = \min_{x \in \Lambda \smallsetminus \{0\}} N(x)$.
The *sphere of* $\Lambda$ is $S = \{x \in \Lambda \mid N(x) = m\}$. Let $s = \frac{|S|}{2}$ ($2s$ is the *kissing number* of $\Lambda$).
The *Gram matrix* of a given basis $\mathcal{B} = (\mathbf{e_1}, \dots, \mathbf{e_n})$ for $\Lambda$ is $\mathrm{Gram}(\mathcal{B}) = \big(e_i \cdot e_j\big)$. Let $\det(\Lambda) = \det(\mathrm{Gram}(\mathcal{B}))$.

The density of the sphere packing attached to $\Lambda$ is proportional to $\gamma(\Lambda)^{n/2}$ where

$$\gamma(\Lambda) = \frac{\min \Lambda}{\det(\Lambda)^{1/n}}$$

is the *Hermite invariant of* $\Lambda$.

Dual version (A-MB +JM):

$$\gamma'(\Lambda) = \big(\gamma(\Lambda) \cdot \gamma(\Lambda^*)\big)^{1/2} = \big((\min \Lambda) \min(\Lambda^*)\big)^{1/2}.$$

Here, $\Lambda^*$ is the *dual lattice to* $\Lambda$, namely

$$\Lambda^* = \{x \in E \mid \forall\, y \in \Lambda,\, x \cdot y \in \mathbb{Z}\}.$$

## Extreme lattices (I)

Formal definitions in the space $\mathrm{End}^s(E)$ of symmetric endomorphisms; for non-zero $x \in E$, $p_x$ stands for the orthogonal projection onto the line $\mathbb{R}\,x$:

- $\Lambda$ is *perfect* if the $p_x,\ x \in S$ span $\mathrm{End}^s(E)$ ;
- $\Lambda$ is *weakly eutactic* if there is a relation $\mathrm{Id} = \sum_{x\in S} \rho_x\, p_x$ with real coefficients $\rho_x$.
- $\Lambda$ is *eutactic* if there is a relation with strictly positive coefficients $\rho_x$.
- $\Lambda$ is *strongly eutactic* if there is a relation with equal (strictly positive) coefficients $\rho_x$.

**Remark 3** If there exists a relation with rational $\rho_x$, $\Lambda$ is *rational*, i.e. proportional to an integral lattice.

## Extreme lattices (II)

**Theorem 2** *(*Korkine & Zolotareff, 1877)

1. *"Extreme"* $\Longrightarrow$ *"Perfect".*
2. *"Perfect"* $\Longrightarrow$ *"Rational".*

**Theorem 3** *(*Voronoï, 1907)

*"Extreme"* $\Longleftrightarrow$ *"Perfect"* + *"Eutactic".*

**Theorem 4** (A-MB & JM;
Voronoï for perfect lattices;
Avner Ash for eutactic lattices)
*In a given dimension, there are only finitely many weakly eutactic lattices (up to similarity).*

**Problem** Classify the weakly eutactic lattices in a given dimension.

**Known results**:

- $n \leq 4$: Štogrin, 1974; A-MB + JM, 1996.
- $n = 5$: Batut, Math. Comp., 2001.

## Venkov's theory (I)

Evaluating $\sum_{x \in S} p_x$ on a basis, one immediately recognizes the notion of strong eutaxy. Hence:

**Proposition**

"$\Lambda$ is strongly eutactic" $\iff$ "$S(\Lambda)$ is a $2$-design".

**Definition** $\Lambda$ is *strongly perfect* if $S(\Lambda)$ is a $5$-design.

**Theorem 5** (Venkov) *A strongly perfect lattice $\Lambda$ is extreme.*

Since a $t$-design is a $t'$-design for all $t' \leq t$, the finiteness theorem for weakly eutactic lattices implies that given $t \geq 2$ and $n$, there are only finitely many $n$-dimensional strongly perfect lattices. **Classification** ?

**Remark 4** Up to dimension $5$, the weakly, hence also the strongly eutactic lattices have been classified. No direct procedure is available.

## Venkov's theory (II)

The two basic identities for 4-designs read

$$\sum_{x\in S/\{\pm 1\}} (x\cdot\alpha)^2 = \frac{s}{n}\,(\min\Lambda)\,N(\alpha)\,;$$
$$\sum_{x\in S/\{\pm 1\}} (x\cdot\alpha)^4 = \frac{3s}{n(n+2)}\,(\min\Lambda)^2\,N(\alpha)^2\,.$$

**Consequences**.

**(1)** "$\Lambda$ strongly perfect" $\Longrightarrow \gamma'(\Lambda) \geq \dfrac{n+2}{3}$.

[For 6-designs, the inequality is strict.]

**(2)** $\Lambda$ integral of minimum $m \geq 2 \Longrightarrow n \leq 3(m^2-1)$.

## Low dimensions and root lattices

The known classification of perfect lattices in dimension $n \le 7$ together with the upper bound for $\gamma'$ immediately show:

**Theorem 6** *Up to similarity, the strongly perfect lattices in dimension* $n \le 7$ *are*

$$\mathbb{Z},\ \mathbb{A}_2,\ \mathbb{D}_4,\ \mathbb{E}_6,\ \mathbb{E}_6^*,\ \mathbb{E}_7,\ \mathbb{E}_7^*\,.$$

For root lattices (integral lattices generated by vectors of norm $1$ or $2$) and their duals, just add $\mathbb{E}_8$ to this list.

## Other classification results

Dimension $8$ – $11$ (Venkov, Nebe–V.). $\mathbb{E}_8$, $K'_{10}$, ${K'_{10}}^*$.

Minimum $3$ (Venkov). $\sqrt{3}\,\mathbb{Z}$, $\sqrt{2}\,\mathbb{E}_7^*$, $\mathrm{O}_{16}$, $\mathrm{O}_{22}$, $\mathrm{O}_{23}$.

Minimum $m \leq 5$, $7$-designs (J.M.).

$\mathbb{Z}$, $\mathbb{E}_8$, $\mathrm{O}_{23}$ (the shorter Leech lattice, of minimum $3$). $\Lambda_{16}$ (the Barnes-Wall lattice), $\Lambda_{23}$, $\Lambda_{24}$ (the Leech lattice), and the even unimodular lattices of minimum $4$ and dimension $32$; minimum $5$ does not occur.

**Remark 5** Let $\Lambda$ of dimension $n \geq 2$, and let $t$ be the largest even integer such that $\Lambda$ is a $t$-design. Lattices are known for which $t = 0, 2, 4, 6, 10$.

**Questions**. Are there lattices with $t = 8$ or $t \geq 11$? With $t = 10$ which *are not* even–unimodular of dimension $n \equiv 0 \mod 24$?

## Modular lattices (I)

Let $\ell$ be a positive integer. We say that $\Lambda$ is *$\ell$-modular* if it is integral, and if there exists a similarity with multiplier $\ell$ which maps $\Lambda^*$ onto $\Lambda$. We restrict ourselves to *even* lattices and suppose that $\ell$ is a prime s. t. $(\ell + 1) \mid 24$ (or $\ell = 1$). Work of Quebbemann, relying on the fact that the theta series of $\Lambda$ is modular for the *Fricke group* of level $\ell$ (twice larger than $\Gamma_0(\ell)$), then shows the upper bound

$$\min \Lambda \leq 2 + \left\lfloor \frac{n(\ell+1)}{48} \right\rfloor .$$

Lattices whose minimum meets this bound are called *extremal*.

**Warning.** **Extremal is not extreme**. However ...

**Remark 6** *The dimension of an $\ell$-modular lattice satisfies the congruence $n \equiv 0 \bmod 2$, and even $n \equiv 0 \bmod 4$ if $\ell = 2$ and $n \equiv 0 \bmod 8$ if $\ell = 1$.*

## Modular lattices (II)

Applying the theory of modular forms with harmonic coefficients, Christine Bachoc and Boris Venkov proved the following results (which indeed are valid for *all* layers):

**(a) Strong perfection**.

$\ell = 1,\ n \equiv 0 \bmod 24$: 11-design.

$\ell = 1,\ n \equiv 8 \bmod 24$; $\ell = 2,\ n \equiv 0 \bmod 16$: 7-design.

$\ell = 2,\ n \equiv 4 \bmod 16$; $\ell = 3,\ n \equiv 0$ or $2 \bmod 12$;
$\ell = 5,\ n = 16$: 5-design.

**(b) Strong eutaxy**.

$\ell = 1,\ n \equiv 16 \bmod 24$; $\ell = 2,\ n \equiv 8 \bmod 16$; $\ell = 3,\ n \equiv 4$ or $6 \bmod 12$; $\ell = 5,\ n \equiv 0 \bmod 8$; $\ell = 7,\ n \equiv 0 \bmod 6$.

## THE BARNES-WALL SERIES

Given $\Lambda$ integral and primitive, and $\sigma \in \mathrm{Aut}(\Lambda)$ with $\sigma^2 = -\mathrm{Id}$, define a $2n$-dimensional lattice by

$$\Lambda' = \{(x,y) \in \Lambda \times \Lambda \mid y \equiv \sigma x \mod 2\Lambda\}\,.$$

Applying inductively this construction and rescaling conveniently the resulting lattices, we define an infinite series of integral and primitive lattices, whose minima double every two steps.
When $\Lambda$ is unimodular, these lattices are alternatively $1$- and $2$-modular.

Starting from $\Lambda = \mathbb{Z}^2$ and $\sigma(x,y) = (-y,x)$, we obtain the *Barnes-Wall series* $\mathrm{BW}_{2^n}$: $\mathbb{D}_4$, $\mathbb{E}_8$, $\Lambda_{16}$, ..., of minima $2, 2, 4, 4, 8, 8, \ldots$. Using the description of their minimal vectors in terms of the *Reed-Muller codes*, VENKOV has proved:

**Theorem 7** *From $n = 8$ onwards, $S(\mathrm{BW}_{2^n})$ is a $7$-design.*

[Probably, all layers are $7$-designs; this would be a consequence of a slight improvement of results by SIDEL'NIKOV's in invariant theory.]

**Known strongly perfect lattices, I** ($1 \leq n \leq 19$).

| dim | nom | det | $s$ | $m$ | $s^*$ | $m^*$ | Type | Rem. |
|---|---|---|---|---|---|---|---|---|
| 1 | $\mathbb{Z}$ | 1 | 1 | 1 | 1 | 1 | min. | $1-$ mod. |
| 2 | $\mathbb{A}_2$ | 3 | 3 | 2 | 3 | 2 | min. | $3-$ mod. |
| 4 | $\mathbb{D}_4$ | 4 | 12 | 2 | 12 | 2 | min. | $2-$ mod. |
| 6 | $\mathbb{E}_6$ | 3 | 36 | 2 | 27 | 4 | min. | |
| 7 | $\mathbb{E}_7$ | 2 | 63 | 2 | 28 | 3 | min. | $\Lambda^*$ equiang. |
| 8 | $\mathbb{E}_8$ | 1 | 120 | 2 | 120 | 2 | gen. | $1-$ mod. |
| 10 | $K'_{10}$ | 972 | 135 | 4 | 120 | 6 | min. | |
| 12 | $K_{12}$ | 729 | 378 | 4 | 378 | 4 | gen. | $3-$ mod. |
| 14 | $Q_{14}$ | 2187 | 378 | 4 | 378 | 4 | min. | $3-$ mod. |
| 16 | $\Lambda_{16}$ | 256 | 2160 | 4 | 2160 | 4 | gen. | $2-$ mod. |
| $-$ | $O_{16}$ | 64 | 256 | 3 | 1008 | 4 | min. | |
| $-$ | $N_{16}$ | 390625 | 1200 | 6 | 1200 | 6 | gen. | $5-$ mod. |
| 18 | $K'_{18}$ | 243 | 3240 | 4 | 1080 | 6 | gen. | |

**Known strongly perfect lattices, II** ($20 \leq n \leq 24$).

| dim | nom | det | $s$ | $m$ | $s^*$ | $m^*$ | Type | Rem. |
|---|---|---|---|---|---|---|---|---|
| 20 | $N_{20},',''$ | 1024 | 1980 | 4 | 1980 | 4 | gen. | $2-$ mod. |
| 21 | $K'_{21}$ | 36 | 13041 | 4 | 112 | 27 | gen. | ${K'}^*_{21}$ non f.p. |
| 22 | $\Lambda_{22}$ | 12 | 24948 | 4 | 891 | 16 | gen. | |
| 22 | $\Lambda_{22[2]}$ | $2^{20}.3$ | 4224 | 6 | 891 | 8 | min | |
| – | $O_{22}$ | 3 | 1408 | 3 | 891 | 8 | min. | |
| – | $M_{22}$ | 15 | 22275 | 4 | 275 | 36 | gen. | |
| – | $M_{22}[5]$ | $3^{21}.5$ | 7128 | 10 | 275 | 12 | min. | |
| 23 | $\Lambda_{23}$ | 4 | 46575 | 4 | 2300 | 12 | gen. | |
| – | $O_{23}$ | 1 | 2300 | 3 | 2300 | 3 | gen. | $1-$ mod. |
| – | $M_{23}$ | 6 | 37950 | 4 | 276 | 15 | gen. | $\Lambda^*$ equiang. |
| – | $M_{23}[2]$ | $2.3^{22}$ | 11178 | 10 | 276 | 5 | min. | $\Lambda^*$ equiang. |
| 24 | $\Lambda_{24}$ | 1 | 98280 | 4 | 98280 | 4 | gen. | $1-$ mod. |
| – | $N_{24}$ | $3^{12}$ | 13104 | 6 | 13104 | 6 | gen. | $3-$ mod. |

## Table for minimum 3

Lower and upper bounds for $s_3(n)$, $n \le 24$.

| $n$ | **1** | 2 | 3 | 4 | 5 | 6 | **7** | 8 |
|---|---|---|---|---|---|---|---|---|
| $s_3(n) \ge$ | **1** | 2 | 4 | 6 | 10 | 16 | **28** | 30 |
| $s_3(n) \le$ | **1** | 2 | 4 | 6 | 10 | 16 | **28** | 30 |
| $n$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 | **16** |
| $s_3(n) \ge$ | 34 | 40 | 52 | 68 | 88 | 112 | 160 | **256** |
| $s_3(n) \le$ | 34 | 63 | 81 | 103 | 129 | 162 | 203 | **256** |
| $n$ | 17 | 18 | 19 | 20 | 21 | **22** | **23** | 24 |
| $s_3(n) \ge$ | 288 | 352 | 448 | 640 | 896 | **1408** | **2300** | 2301 |
| $s_3(n) \le$ | 322 | 411 | 531 | 703 | 965 | **1408** | **2300** | 4991 |

Except for $n = 8, 9$, the proof relies on the theory of spherical designs, which gives at once the results for dimensions

$23$; $1$ and $22 = 23 - 1$; $7$ and $16 = 23 - 7$.

The exact values found in dimensions $n \le 7$ can be widely extended; **see next slide** $\longrightarrow \cdots \cdots \cdots \longrightarrow$

## ODD MINIMUM

Let $s_m(n)$ be the maximum of $s$ on primitive, integral lattices of minimum $m$.

**Theorem 8** *The values of* $s_m(n)$ *for* $m \geq 3$ *odd and* $n \leq 7$ *are:*

| **n** | 1 | 2 | 3 | 4 | 5 | 6 | **7** | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{m = 3}$ | 1 | 2 | 4 | 6 | 10 | 16 | **28** | 30 | 34 |
| $\mathbf{m \geq 5}$ | 1 | 2 | 4 | 6 | 10 | 16 | **27** | 30 ? | 34 ? |
| (*general*) | 1 | 3 | 6 | 12 | 20 | 36 | 63 | 120 | 136 |

Up to $n = 7$, the numbers in the first line are upper bounds for $s$ which hold for any lattice having no hexagonal section with the same minimum (WATSON, 1972); these bounds are attained on convenient cross–sections of $\sqrt{2}\,\mathbb{E}_7^*$, of minimum $3$.

For odd $m \geq 5$, consider the Voronoï path $\mathbb{E}_7^*$ —— $\mathbb{E}_7$.

## MINIMUM 3, $n \leq 9$

For these dimensions, we have obtained fairly precise classification results as far as only large values of $s$ are concerned.

$n = 5$. $s = 10$**,** $s = 8$ **(2),** $s = 7$ **(4)**.

$n = 6$. $s = 16$**,** $s = 12$**,** $s = 11$ **(2),** $s = 10$ **(5)**.

$n = 7$. $s = 28$**,** $s = 18$**,** $s = 17$ **(2),** $s = 16$ **(2) or** $s \leq 14$.

$n = 8$. $s = 30$**,** $s = 29$**,** $s = 22$ $s = 20$ **(7),** $s = 19$ **(5)**.

$n = 9$. $s = 34$**,** $s = 32$ **(3),** $s = 31$ **(2?),** $s = 30$ **(2?), or** $s \leq 28$ **(?)**.

[$n = 10$. **(??)** $s = 40$ **(4), or** $s \leq 38$.]

[$m = 5$, $n = 7$. $s = 27$ **(1) or (?)** $s \leq 21$.]

## Proofs for minimum 3

Our first task was to bound $\min \Lambda^*$ for integral, well-rounded lattices of minimum $3$. The exact bounds are not known for $n \geq 8$. We have $\min \Lambda^* \mathbf{=1}$ if $n = \mathbf{7}$, $\min \Lambda^* < 1$ if $n \leq 8, n \neq \mathbf{7}$, $\min \Lambda^* < \frac{4}{3}$ if $n = 9$ (for $\min \Lambda^* \leq 1$ expected).

Using these bounds, we were able to bound the number of minimal vectors outside a hyperplane section and thus use induction; these bounds were obtained by constructing auxiliary root systems.

We also used a detail study of the index of a well-rounded sublattice having the same minimum, considering separately "high" and "low" indices.