COMBINATORICS AND LATTICES

A few years ago, Boris VENKOV discovered that there are some interesting connections between the relatively recent theory of **spherical designs** on the one hand, and the theory of **extreme lattices** initiated by Alexandre KORKINE and Igor ZOLOTAREFF in their 1877 paper and developed thirty years later by Georges VORONOÏ.

A *lattice* is a discrete subgroup of a Euclidean space E, of maximal rank, indeed $n = \dim E$. A lattice Λ is *extreme* if the density of the sphere packing canonically attached to any lattice attains a local maximum at Λ.

Our spherical designs will live on the sphere of minimal vectors of a lattice; more generally, we shall sometimes consider the various layers; minimal vectors in the **dual lattice** will often play an important rôle.

References

Basic book: Réseaux euclidiens, designs sphériques et groupes, L'Enseignement Mathématique, Monographie **37**, J. Martinet, ed., Genève, 2001; see in particular the contributions of Venkov, Bachoc–Venkov, Martinet, Martinet–Venkov.

Further related papers:

Gabi Nebe–Venkov, *The strongly perfect lattices in dimension* 10, J. TdN Bx, **12** (2000), 503–518.

A.-M. Bergé–Martinet, *Symmetric Groups and Lattices*, Monatshefte Math. (2003), to appear.

Martinet–Venkov, *On integral lattices having an odd minimum*, preprint, 42. pp.

The notion of a spherical design

Let S^{n-1} be the unit sphere with center O, endowed with the standard measure scaled to volume 1, let $t > 0$ be an integer, and let $X \subset \Sigma$ be a finite set.

We say that X is a (*spherical*) **t-***design* if

$$\int_{\Sigma} f\, dx = \frac{1}{|X|} \sum_{x \in X} f(x)$$

holds for all polynomials of degree at most **t** on E.

Equivalent definition: the integral above is zero for all *homogeneous, harmonic polynomials* of degree at most **t**.

Example 1 *"X is a 1-design"* $\Longleftrightarrow$ *"0 is the center of gravity of X".*

Remark 1 Any symmetric set which is a $2t$-design is a $(2t+1)$-design.

Remark 2 If $n = 1$, every 2-design is a t-design for all t.

Design identities

From now on, all designs are symmetric.

Theorem 1 *If* $n \geq 2$ *and if* $t \geq 2$ *is* even, *the following conditions are equivalent:*

1. X *is a* t*-design.*
2. *For all even* $p \leq t$*, there exists a constant* c_p *such that for all* $\alpha \in E$*,*

$$\sum_{x \in X} (x \cdot \alpha)^p = c_p (\alpha \cdot \alpha)^{p/2} (x \cdot x)^{p/2}.$$

3. *The identity above holds for* $p = t$*.*

Moreover, when these conditions hold, we have

$$c_p = \frac{1.3.5 \ldots (p-1)}{n(n+2)\ldots(n+p-2)} |X| .$$

[However, to consider all even integers $p \leq t$ may prove useful.]

Some notation for lattices

The *norm of* $x \in E$ is $N(x) = x \cdot x$.
The *minimum of* Λ is $m = \min_{x \in \Lambda \smallsetminus \{0\}} N(x)$.
The *sphere of* Λ is $S = \{x \in \Lambda \mid N(x) = m\}$. Let $s = \frac{|S|}{2}$ ($2s$ is the *kissing number* of Λ).
The *Gram matrix* of a given basis $\mathcal{B} = (\mathbf{e_1}, \ldots, \mathbf{e_n})$ for Λ is $\mathrm{Gram}(\mathcal{B}) = \big(e_i \cdot e_j\big)$. Let $\det(\Lambda) = \det(\mathrm{Gram}(\mathcal{B}))$.

The density of the sphere packing attached to Λ is proportional to $\gamma(\Lambda)^{n/2}$ where

$$\gamma(\Lambda) = \frac{\min \Lambda}{\det(\Lambda)^{1/n}}$$

is the *Hermite invariant of* Λ.

Dual version (A-MB +JM):

$$\gamma'(\Lambda) = \big(\gamma(\Lambda) \cdot \gamma(\Lambda^*)\big)^{1/2} = \big((\min \Lambda) \min(\Lambda^*)\big)^{1/2}.$$

Here, Λ^* is the *dual lattice to* Λ, namely

$$\Lambda^* = \{x \in E \mid \forall\, y \in \Lambda,\ x \cdot y \in \mathbb{Z}\}.$$

Extreme lattices (I)

Formal definitions in the space $\mathrm{End}^s(E)$ of symmetric endomorphisms; for non-zero $x \in E$, p_x stands for the orthogonal projection onto the line $\mathbb{R}\,x$:

- Λ is *perfect* if the $p_x,\ x \in S$ span $\mathrm{End}^s(E)$;
- Λ is *weakly eutactic* if there is a relation $\mathrm{Id} = \sum_{x\in S} \rho_x\, p_x$ with real coefficients ρ_x.
- Λ is *eutactic* if there is a relation with strictly positive coefficients ρ_x.
- Λ is *strongly eutactic* if there is a relation with equal (strictly positive) coefficients ρ_x.

Remark 3 If there exists a relation with rational ρ_x, Λ is *rational*, i.e. proportional to an integral lattice.

Extreme lattices (II)

Theorem 2 (Korkine & Zolotareff, 1877)

1. *"Extreme"* $\Longrightarrow$ *"Perfect".*
2. *"Perfect"* $\Longrightarrow$ *"Rational".*

Theorem 3 (Voronoï, 1907)

"Extreme" $\Longleftrightarrow$ *"Perfect" + "Eutactic".*

Theorem 4 (A-MB & JM;
Voronoï for perfect lattices;
Avner Ash for eutactic lattices)
In a given dimension, there are only finitely many weakly eutactic lattices (up to similarity).

Problem Classify the weakly eutactic lattices in a given dimension.

Known results:

- $n \leq 4$: Štogrin, 1974; A-MB + JM, 1996.
- $n = 5$: Batut, Math. Comp., 2001.

Venkov's theory (I)

Evaluating $\sum_{x \in S} p_x$ on a basis, one immediately recognizes the notion of strong eutaxy. Hence:

Proposition

"Λ is strongly eutactic" $\iff$ "$S(\Lambda)$ is a 2-design".

Definition Λ is *strongly perfect* if $S(\Lambda)$ is a 5-design.

Theorem 5 (Venkov) *A strongly perfect lattice Λ is extreme.*

Since a t-design is a t'-design for all $t' \leq t$, the finiteness theorem for weakly eutactic lattices implies that given $t \geq 2$ and n, there are only finitely many n-dimensional strongly perfect lattices. **Classification** ?

Remark 4 Up to dimension 5, the weakly, hence also the strongly eutactic lattices have been classified. No direct procedure is available.

Venkov's theory (II)

The two basic identities for 4-designs read

$$\sum_{x \in S/\{\pm 1\}} (x \cdot \alpha)^2 = \frac{s}{n} (\min \Lambda)\, N(\alpha)\,;$$

$$\sum_{x \in S/\{\pm 1\}} (x \cdot \alpha)^4 = \frac{3s}{n(n+2)} (\min \Lambda)^2\, N(\alpha)^2\,.$$

Consequences.

(1) "Λ strongly perfect" $\Longrightarrow \gamma'(\Lambda) \geq \dfrac{n+2}{3}\,.$

[For 6-designs, the inequality is strict.]

(2) Λ integral of minimum $m \geq 2 \Longrightarrow n \leq 3(m^2-1)$.

LOW DIMENSIONS AND ROOT LATTICES

The known classification of perfect lattices in dimension $n \leq 7$ together with the upper bound for γ' immediately show:

Theorem 6 *Up to similarity, the strongly perfect lattices in dimension* $n \leq 7$ *are*

$$\mathbb{Z},\ \mathbb{A}_2,\ \mathbb{D}_4,\ \mathbb{E}_6,\ \mathbb{E}_6^*,\ \mathbb{E}_7,\ \mathbb{E}_7^*\ .$$

For root lattices (integral lattices generated by vectors of norm 1 or 2) and their duals, just add $\mathbb{E}_8$ to this list.

Other classification results

Dimension $8 - 11$ (Venkov, Nebe–V.). $\mathbb{E}_8$, K'_{10}, ${K'_{10}}^*$.

Minimum 3 (Venkov). $\sqrt{3}\,\mathbb{Z}$, $\sqrt{2}\,\mathbb{E}_7^*$, O_{16}, O_{22}, O_{23}.

Minimum $m \leq 5$, 7-designs (J.M.).

$\mathbb{Z}$, $\mathbb{E}_8$, O_{23} (the shorter Leech lattice, of minimum 3). Λ_{16} (the Barnes-Wall lattice), Λ_{23}, Λ_{24} (the Leech lattice), and the even unimodular lattices of minimum 4 and dimension 32; minimum 5 does not occur.

Remark 5 Let Λ of dimension $n \geq 2$, and let t be the largest even integer such that Λ is a t-design. Lattices are known for which $t = 0, 2, 4, 6, 10$.

Questions. Are there lattices with $t = 8$ or $t \geq 11$? With $t = 10$ which *are not* even–unimodular of dimension $n \equiv 0 \bmod 24$?

Modular lattices (I)

Let ℓ be a positive integer. We say that Λ is ℓ-*modular* if it is integral, and if there exists a similarity with multiplier ℓ which maps Λ^* onto Λ. We restrict ourselves to *even* lattices and suppose that ℓ is a prime s. t. $(\ell+1) \mid 24$ (or $\ell = 1$). Work of Quebbemann, relying on the fact that the theta series of Λ is modular for the *Fricke group* of level ℓ (twice larger than $\Gamma_0(\ell)$), then shows the upper bound

$$\min \Lambda \leq 2 + \left\lfloor \frac{n(\ell+1)}{48} \right\rfloor .$$

Lattices whose minimum meets this bound are called *extremal*.

Warning. **Extremal is not extreme**. However ...

Remark 6 *The dimension of an ℓ-modular lattice satisfies the congruence $n \equiv 0 \bmod 2$, and even $n \equiv 0 \bmod 4$ if $\ell = 2$ and $n \equiv 0 \bmod 8$ if $\ell = 1$.*

Modular lattices (II)

Applying the theory of modular forms with harmonic coefficients, Christine Bachoc and Boris Venkov proved the following results (which indeed are valid for *all* layers):

(a) Strong perfection.

$\ell = 1,\ n \equiv 0 \bmod 24$: 11-design.

$\ell = 1,\ n \equiv 8 \bmod 24$; $\ell = 2,\ n \equiv 0 \bmod 16$: 7-design.

$\ell = 2,\ n \equiv 4 \bmod 16$; $\ell = 3,\ n \equiv 0$ or $2 \bmod 12$; $\ell = 5,\ n = 16$: 5-design.

(b) Strong eutaxy.

$\ell = 1,\ n \equiv 16 \bmod 24$; $\ell = 2,\ n \equiv 8 \bmod 16$; $\ell = 3,\ n \equiv 4$ or $6 \bmod 12$; $\ell = 5,\ n \equiv 0 \bmod 8$; $\ell = 7,\ n \equiv 0 \bmod 6$.

The Barnes-Wall series

Given Λ integral and primitive, and $\sigma \in \mathrm{Aut}(\Lambda)$ with $\sigma^2 = -\mathrm{Id}$, define a $2n$-dimensional lattice by

$$\Lambda' = \{(x, y) \in \Lambda \times \Lambda \mid y \equiv \sigma x \mod 2\Lambda\}\,.$$

Applying inductively this construction and rescaling conveniently the resulting lattices, we define an infinite series of integral and primitive lattices, whose minima double every two steps.
When Λ is unimodular, these lattices are alternatively 1- and 2-modular.

Starting from $\Lambda = \mathbb{Z}^2$ and $\sigma(x, y) = (-y, x)$, we obtain the *Barnes-Wall series* BW_{2^n}: $\mathbb{D}_4$, $\mathbb{E}_8$, Λ_{16}, ..., of minima $2, 2, 4, 4, 8, 8,$ .... Using the description of their minimal vectors in terms of the *Reed-Muller codes*, Venkov has proved:

Theorem 7 *From $n = 8$ onwards, $S(\mathrm{BW}_{2^n})$ is a 7-design.*
[Probably, all layers are 7-designs; this would be a consequence of a slight improvement of results by Sinelnikov's in invariant theory.]

Known strongly perfect lattices, I ($1 \leq n \leq 19$).

dim	nom	det	s	m	s^*	m^*	Type	Rem.
1	$\mathbb{Z}$	1	1	1	1	1	min.	$1-$ mod.
2	$\mathbb{A}_2$	3	3	2	3	2	min.	$3-$ mod.
4	$\mathbb{D}_4$	4	12	2	12	2	min.	$2-$ mod.
6	$\mathbb{E}_6$	3	36	2	27	4	min.	
7	$\mathbb{E}_7$	2	63	2	28	3	min.	Λ^* equiang.
8	$\mathbb{E}_8$	1	120	2	120	2	gen.	$1-$ mod.
10	K'_{10}	972	135	4	120	6	min.	
12	K_{12}	729	378	4	378	4	gen.	$3-$ mod.
14	Q_{14}	2187	378	4	378	4	min.	$3-$ mod.
16	Λ_{16}	256	2160	4	2160	4	gen.	$2-$ mod.
$-$	O_{16}	64	256	3	1008	4	min.	
$-$	N_{16}	390625	1200	6	1200	6	gen.	$5-$ mod.
18	K'_{18}	243	3240	4	1080	6	gen.	

Known strongly perfect lattices, II ($20 \leq n \leq 24$).

dim	nom	det	s	m	s^*	m^*	Type	Rem.
20	$N_{20},',''$	1024	1980	4	1980	4	gen.	$2-\text{mod.}$
21	K'_{21}	36	13041	4	112	27	gen.	${K'}^*_{21}$ non f.p.
22	Λ_{22}	12	24948	4	891	16	gen.	
22	$\Lambda_{22[2]}$	$2^{20}.3$	4224	6	891	8	min	
–	O_{22}	3	1408	3	891	8	min.	
–	M_{22}	15	22275	4	275	36	gen.	
–	$M_{22}[5]$	$3^{21}.5$	7128	10	275	12	min.	
23	Λ_{23}	4	46575	4	2300	12	gen.	
–	O_{23}	1	2300	3	2300	3	gen.	$1-\text{mod.}$
–	M_{23}	6	37950	4	276	15	gen.	Λ^* equiang.
–	$M_{23}[2]$	2.3^{22}	11178	10	276	5	min.	Λ^* equiang.
24	Λ_{24}	1	98280	4	98280	4	gen.	$1-\text{mod.}$
–	N_{24}	3^{12}	13104	6	13104	6	gen.	$3-\text{mod.}$

Table for minimum 3

Lower and upper bounds for $s_3(n)$, $n \le 24$.

n	**1**	2	3	4	5	6	**7**	8
$s_3(n) \ge$	**1**	2	4	6	10	16	**28**	30
$s_3(n) \le$	**1**	2	4	6	10	16	**28**	30
n	9	10	11	12	13	14	15	**16**
$s_3(n) \ge$	34	40	52	68	88	112	160	**256**
$s_3(n) \le$	34	63	81	103	129	162	203	**256**
n	17	18	19	20	21	**22**	**23**	24
$s_3(n) \ge$	288	352	448	640	896	**1408**	**2300**	2301
$s_3(n) \le$	322	411	531	703	965	**1408**	**2300**	4991

Except for $n = 8, 9$, the proof relies on the theory of spherical designs, which gives at once the results for dimensions

$$23;\ 1 \text{ and } 22 = 23 - 1;\ 7 \text{ and } 16 = 23 - 7.$$

The exact values found in dimensions $n \le 7$ can be widely extended; **see next slide** $\longrightarrow \cdots\ \cdots\ \cdots\ \longrightarrow$

Odd minimum

Let $s_m(n)$ be the maximum of s on primitive, integral lattices of minimum m.

Theorem 8 *The values of $s_m(n)$ for $m \geq 3$ odd and $n \leq 7$ are:*

n	1	2	3	4	5	6	**7**	8	9
$\mathbf{m = 3}$	1	2	4	6	10	16	**28**	30	34
$\mathbf{m \geq 5}$	1	2	4	6	10	16	**27**	30 ?	34 ?
$(general)$	1	3	6	12	20	36	63	120	136

Up to $n = 7$, the numbers in the first line are upper bounds for s which hold for any lattice having no hexagonal section with the same minimum (Watson, 1972); these bounds are attained on convenient cross–sections of $\sqrt{2}\,\mathbb{E}_7^*$, of minimum 3.

For odd $m \geq 5$, consider the Voronoï path $\mathbb{E}_7^*$ —— $\mathbb{E}_7$.

Minimum 3, $n \leq 9$

For these dimensions, we have obtained fairly precise classification results as far as only large values of s are concerned.

$\underline{n = 5}$. $s = 10$**,** $s = 8$ **(2),** $s = 7$ **(4)**.

$\underline{n = 6}$. $s = 16$**,** $s = 12$**,** $s = 11$ **(2),** $s = 10$ **(5)**.

$\underline{n = 7}$. $s = 28$**,** $s = 18$**,** $s = 17$ **(2),** $s = 16$ **(2) or** $s \leq 14$.

$\underline{n = 8}$. $s = 30$**,** $s = 29$**,** $s = 22$ $s = 20$ **(7),** $s = 19$ **(5)**.

$\underline{n = 9}$. $s = 34$**,** $s = 32$ **(3),** $s = 31$ **(2?),** $s = 30$ **(2?), or** $s \leq 28$ **(?)**.

[$\underline{n = 10}$. **(??)** $s = 40$ **(4), or** $s \leq 38$.]

[$\underline{m = 5,\ n = 7}$. $s = 27$ **(1) or (?)** $s \leq 21$.]

Proofs for minimum 3

Our first task was to bound $\min \Lambda^*$ for integral, well-rounded lattices of minimum 3. The exact bounds are not known for $n \geq 8$. We have $\min \Lambda^* \mathbf{=1}$ if $n = 7$, $\min \Lambda^* < 1$ if $n \leq 8$, $n \neq 7$, $\min \Lambda^* < \frac{4}{3}$ if $n = 9$ (for $\min \Lambda^* \leq 1$ expected).

Using these bounds, we were able to bound the number of minimal vectors outside a hyperplane section and thus use induction; these bounds were obtained by constructing auxiliary root systems.

We also used a detail study of the index of a well-rounded sublattice having the same minimum, considering separately “high” and “low” indices.