J. Martinet January 31st, 2011; corrected on April 5th, 2014

Comments on the Ph D theses of
J.-L. Baril, M. Laïhem and H.Napias

Here are some comments on the three Bordeaux Theses of Mohamed Laïhem (December 4th, 1992), Jean-Luc Baril and Huguette Napias (January 25th, 1996), that can be downloaded from my homepage, under the link *publications by various authors*; the thesis advisers where A.-M. Bergé for the first two and myself for the third one. All three theses are dedicated to the study of perfect (Euclidean) lattices and related notions.
The references are those of my homepage, cited [homeMar] and those to be found on the link *recent publications* of [homeMar], *Corrected and extended reference list of the book "Perfect Lattices in Euclidean Spaces"*, quoted here [Mar]. The theses referred to in the title are [Lah], [Bari], and [Nap2].

1. M. L. The aim of Laïhem's thesis was to construct all perfect lattices having a perfect hyperplane section with the same minimum. There are 33 perfect, 7-dimensional lattices, among which the three root lattice $\mathbb{A}_7$, $\mathbb{D}_7$, $\mathbb{E}_7$. Laïem classified those which contain one of the remaining 30 lattices, producing a list of 1171 perfect lattices, and quoted 4 more lattices, but did not prove that these four lattices where the only perfect lattices having a hyperplane section isometric to a root lattice.

This last result was proved by Baril in the first chapter of his thesis. This proved that there are exactly 1175 perfect, 8-dimensional lattices having a hyperplane section with the same minimum.

The list can be read in [homeMar] (Catalogue of Perfect Lattices) under the name *Laïhem lattices* $lh(i)$, $1 \le i \le 1175$, the last four being the root lattices $\mathbb{E}_8$, $\mathbb{D}_8$, $\mathbb{A}_8$ and *Barnes's lattice* $\mathbb{A}_8^2 = \langle \mathbb{A}_8, \mathbb{A}_7^2 \rangle$ where $\mathbb{A}_7^2 \simeq \mathbb{E}_7$ is a *Coxeter lattice.*

Basically, classifying perfect, n-dimensional lattices having a cross-section by a hyperplane section H which is perfect with the same minimum is done using the following steps. One completes an integral, primitive (symmetric) Gram matrix for the section by a last column $[a_1, ..., a_n]$ where $a_n = m$, the minimum of the section, and the other a_i are rational numbers having small denominators, with $|a_i| \le \frac{m}{2}$. One can guess this way a perfect extension of the section, which we use to perform a Voronoi-like neighbouring algorithm as described in

[Mar], Section 13.6. This algorithm produces all perfect lattices having the given hyperplane section, up to H-*isometries*, i.e. isometries preserving H. There remains then to test for general isometry lattices which have the same standard invariants (minimum, determinant, kissing number,... A few example of isometric, not H-isometric lattices occur in Laïhem's thesis). To get the exact list of Laïhem lattices, there of course remains to to test for isometry lattices with the same invariants found above distinct cross-sections.

2. J.-L. B. Baril's thesis consists of three chapters. In the first one he classifies some perfect lattices having a given section, first in dimension 8 (see above) then in dimensions 9, 10, and 11.

The second chapter is devoted to patchwork constructions (see [Mar], Section 13.6), and in particular to lattices which are direct sum of perfect lattices of dimension 6 (7 lattices) and 2 (the hexagonal lattice $\mathbb{A}_2$). He proves that (always up to similarity) there are exactly 53 such lattices off Laïhem's list, the *Baril lattices*, denoted by $bari(i)$, $1 \leq i \leq 53$.

The third and last chapter is devoted to the *dual Hermite invariant* γ' (alias *Bergé-Martinet invariant*), the geometric mean of the Hermite invariants of a lattice and its dual. He finds lattices having a large invariant γ' on some conveniently chosen Voronoi paths. In particular, he constructs an 9-dimensional, dual-extreme lattice L for which $\gamma'(L)^2 = \frac{16}{5}$, the maximal known value, also attained on the Coxeter lattice $\mathbb{A}_9^2$. More information on this *Baril lattice* can be read in [homeMar], *recent publications*, after Proposition 3.8.C2.

3. H. N. Napias's thesis consists of four chapters. In the first one she describes some lattices contained either in the Leech lattice or in the Bachoc lattice of dimension 32 constructed over the Hurwitz quaternionic maximal order $\mathfrak{M}$.

The second chapter deals with Voronoi's neighbouring procedure, in two situations. In the first (theoretical) one she considers the lattices contiguous to Coxeter lattices $\mathbb{A}_n^r$, $r \mid n+1$, both in the special case $r = \frac{n+1}{2}$ and in the "generic" case $n \geq 9$, $3r \leq n+1$. In the latter case the Voronoi paths start like those of $\mathbb{A}_n$ but the critical value is smaller than that of $\mathbb{A}_n$ (for which the contiguous lattice is $\mathbb{D}_n$). She then considers the previously known 8-dimensional, perfect lattices (essentially, those of Laïhem and Baril), extracts from the list those for which $s = \frac{n+1}{2} = 36$, performs for them the Voronoi algorithm, add the new founded lattices to the previous list, then performs again the Voronoi algorithm for those with $s = 36$, etc. She finally produced a list of 9542 new perfect *Napias lattices*, denoted by $nap(i)$, $1 \leq i \leq 9542$, so that the number of known 8-dimensional, perfect lattices became $1175 + 53 + 9542 = 10770$.

The third chapter is devoted to an analogue of the LLL-algorithm for lattices constructed over various Euclidean rings, namely the rings of integers of the five first imaginary quadratic and maximal orders in the two skew-fields of quaternions over $\mathbb{Q}$ ramified at 2 and 3, respectively. This proved useful to reduce bases of lattices (e.g., the Bachoc lattice referred to above). A joint text with Henri Cohen can be downloaded; see the link.

In the fourth and last chapter, she considers questions related to algebraic number fields K, equipped with the Euclidean structure associated with the trace form $\mathrm{Tr}_{K/\mathbb{Q}}(x\overline{y})$, namely:
(a) does the successive Minkowski minima produce an integral basis ?
(b) In the case when K is cyclic of prime degree $\ell \geq 3$, does the Gauss sum represents the second minimum ? (The first minimum, equal to $[K : \mathbb{Q}]$, is attained exactly on ± 1.) We return below to these questions.

4. Varia.

1 a. Laïhem considered off his thesis the structures of modules over the orders $\mathfrak{O} = \mathbb{Z}[1, i, j, k]$ and $\mathfrak{M} = \langle \mathfrak{O}, \frac{1+i+j+k}{2} \rangle$ (the (maximal) Hurwitz order) inside the field of "usual" quaternions over $\mathbb{Q}$. He noticed that $\mathbb{E}_8$ can be realized as a free module over both $\mathfrak{O}$ and $\mathfrak{M}$ whereas $\mathbb{D}_8$ needs the mixed structure $\mathfrak{O} \oplus \mathfrak{M}$. It was proved later (Plesken, then Coulangeon, unpublished) that torsion free $\mathfrak{O}$-modules of rank m are isomorphic to a direct sum $\mathfrak{O}^r \oplus \mathfrak{M}^{m-r}$. Lattices over $\mathfrak{O}$ are G-lattices (in the sense of [Mar], Section 13.3) for G the quaternion group H_8 and that over $\mathfrak{M}$ are G-lattices for G the double cover $2 \cdot A_4 = \tilde{A}_4$, which contains H_8 to index 3. Classification over $\mathfrak{M}$ is known for $m = 2, 3$ (Sigrist, Schürmann), and implies (Stephanie Vance, [Van1]) that the Barnes-Wall lattice BW_{16} is the densest $\mathfrak{M}$-lattice for $m = 4$. Classification over $\mathfrak{O}$ is not known beyond $m = 2$.

3 a. The list of known 8-dimensional, perfect lattices was extended some months after Napias's work by Batut, who ran the Voronoi algortithm for lattices with kissing numbers $s = 37$ and $s = 38$, obtaining 146 new lattices, denoted by $batu(i)$, $1 \leq i \leq 146$, obtaining a list of 10816 lattices. Dutour Sikirić, Schürmann and Vallentin constructed in 2005 ([D-S-V1]) the Voronoi graph for dimension 8, proving this way that the list above is complete.

It has been checked that all 8-dimensional, perfect lattices possess a hexagonal section with the same minimum, so that the minimum dimension r for which such a lattice has a perfect section with the same minimum belongs to the range $2-7$. One has $r = 7$ for the

1175 lattices, and $r = 6$ holds on 54 lattices, the 53 Baril lattices and one Napias lattice. The lists of lattices for all values of r can now be downloaded from the "catalogue" of my homepage.

3 b. Any lattice of dimension $n \leq 4$ has a basis made of representative of successive minima, so that the problem for number fields occurs only in degrees $n \geq 5$. In the range of Napias's experiments all the fields she considered had integral bases generated by representatives of successive minima. However this is not general, as shown by Bart de Smit who constructed 6-dimensional counter-examples; see *Various publications, On Successive Minima of Rings of Algebraic Integers.* The problem for degree 5 is still open.

3 c. In the range of Napias's experiments, the second minimum for cyclic fields of odd prime degree ℓ is afforded by the Gauss sum. This is true for $\ell = 3$ (two different proofs where given by Napias and by myself). Whether this is general is an open problem from $\ell = 5$ onwards.

More generally the minimum (equal to n) is attained exactly on roots of unity. When the roots of unity do not constitute an integral basis, the problem of the comparison between the next minimum and the Gauss sum is open.