SOME PROBLEMS IN THE THEORY OF EUCLIDEAN LATTICES

JACQUES MARTINET (*)

Abstract. We present here some questions on lattice theory, mainly related to my book *Perfect Lattices in Euclidean Spaces*, cited [M]. These concern kissing numbers, extremality properties, eutaxy, Watson's theory of the index, connections with algebraic number theory and complex analysis, traditional geometry of numbers and diophantine approximations.

Content.
1. References and Notation.
2. Kissing Numbers and Minima. *page 2*
3. Minimal Classes. *page 5*
4. Extremality Properties. *page 7*
5. Eutaxy and Related Properties. *page 13*
6. Watson's Index Theory. *page 15*
7. Lattices Modulo 2. *page 18*
8. Connection with Algebraic Number Theory. *page 19*
9. Analytic and Algebraic Geometry. *page 20*
10. "Classical" Geometry of Numbers. *page 21*
11. Diophantine Approximations. *page 23*

The subjects of the last three sections are somewhat off the scope of [M]. For this reason the problems I consider there are only sketched.

1. References and Notation

References (unless otherwise stated, and except [M] and [M']!) are those of my Springer book [M]: "Grunlehren" nu. 327 or of the *Corrected and extended reference list*, available at

http://jamartin.perso.math.cnrs.fr,

Section "Recent journal publications and preprints",

together with *Erratum to* and *Complements to Perfect Lattices in Euclidean Spaces*, the latter cited [M']. References on pages 1–15 of

2010 *Mathematics Subject Classification.* 11H55, 11H71.
Key words and phrases. Euclidean lattices, real quadratic forms
(*) Jacques.Martinet@math.cnrs.fr . Previous drafts: August, 2006, Febrary, 2012; last update: January, 2015, except for some footnotes.

this reference list are those of the book, references from page 16 onwards are complementary references.

Special references for Sections 10 and 11 are [Cas2], [Cas1], and [C-SwD] and [SwD] below:

[C-SwD] J.W.S. Cassels, H.P.F. Swinnerton-Dyer, *On the product of three homogeneous linear forms and the indefinite ternary quadratic forms*, Philos. Trans. Roy. Soc. London, Series A, **248** (1955), 73–96.

[SwD] H.P.F. Swinnerton-Dyer, *On the product of three homogeneous linear forms*, Acta Arith. **18** (1971), 371–385.

We denote by E a Euclidean space of dimension n (with $n \geq 2$ to avoid trivial situations). For a lattice Λ in E, the notation $S = S(\Lambda)$ (the set of minimal vectors of Λ), $s = \frac{1}{2}|S|$ (the (half-)*kissing number*, $\min \Lambda$ $(= \inf x \cdot x,\ x \in \Lambda \smallsetminus \{0\})$, $\det(\Lambda)$ (the determinant), ... is the traditional notation; see [M], Chapter 1.

Given $x \in E \smallsetminus \{0\}$, we denote by $p_x \in \mathrm{End}^s(E)$ the orthogonal projection to the line $\mathbb{R}x$. The *perfection rank of* Λ, denoted by $r(\Lambda)$ or r, is the dimension of the span in $\mathrm{End}^s(E)$ of the p_x, $x \in S(\Lambda)$. We have $1 \leq r \leq \dim \mathrm{End}^s(E) = \frac{n(n+1)}{2}$, and call *perfection co-rank* the difference $\frac{n(n+1)}{2} - r$. Lattices of perfection co-rank zero are called *perfect*; see [M], Chapter 3.

The *dual lattice to* Λ is

$$\Lambda^* = \{x \in E \mid \forall\, y \in \Lambda,\ x \cdot y \in \mathbb{Z}\}\,.$$

We shall often write s^* for $s(\Lambda^*)$.

The *Hermite invariant of* Λ is $\gamma(\Lambda) = \min \Lambda / \det(\Lambda)^{1/n}$, and the *Hermite constant for dimension* n is $\gamma_n = \max_{\dim \Lambda = n} \gamma(\Lambda)$; note that the density of (the sphere packing attached to) Λ is proportional to $\gamma(\Lambda)^{n/2}$.

Unless otherwise stated, we systematically restrict ourselves in Sections 2 to 6 to *well rounded lattices*, those which contain n independent minimal vectors. This generally suffices for the applications we are going to consider. Moreover, most of the questions we study (though not all) reduce to lower dimensions when lattices which are not well rounded are involved. Note that perfect lattices, weakly eutactic lattices (see Section 5), ... are well-rounded.

2. Kissing Numbers and Minima

2.1. **Kissing numbers in low dimensions.** For a given dimension, the maximum of the kissing number is attained on a perfect lattice (0-dimensional cell), among non-perfect lattices on a Voronoi path (1-dimensional cell), then on a 2-dimensional cell, ...; see Section 3 for the definitions. In his 1971 paper [Wat5], Watson, using the knowledge of the Voronoi graphs for $n \le 6$ (Voronoi for $n \le 5$, Barnes for $n = 6$), was able to find the maximal value of the kissing number up to dimension 9, and to prove various complements which properly belong to the theory of Voronoi graphs in dimensions 7 and 8; for instance, he showed that for $n = 8$, we have $s = 120$ if $\Lambda \sim \mathbb{E}_8$, $s = 75$ on a well-defined Voronoi path connecting two copies of $\mathbb{E}_8$ (along hyperplane sections $\mathbb{E}_7$ and $\mathbb{D}_7$), and $s < 75$ otherwise.

In his 1991 thesis [Ja2], Jaquet, revisiting earlier work of Kaye Stacey, determined the Voronoi graph in dimension 7, and in 2005, Dutour, Schürmann and Vallentin announced the computation of the 8-dimensional Voronoi graph ([D-S-V]).

Problem 2.1. *Using the results above, extend Watson's methods to obtain new information on the kissing number in dimensions $n = 9$ to 11, maybe even $n = 12$.*

Experimental data suggest some precise conjectures. One can show (see [Bari]) that there are two 9-dimensional, perfect lattices having an $\mathbb{E}_8$ section with the same minimum, namely Λ_9, with $s = 136$, and Barnes's $\mathbb{A}_9^3 = \langle \mathbb{E}_8, \mathbb{A}_9 \rangle$, with $s = 129$ (a result from which one could deduce all kissing numbers for lattices as above).

Conjecture 2.2. *Let Λ be a (well-rounded) 9-dimensional lattice. Then either $s(\Lambda) \ge 121$, and Λ has an $\mathbb{E}_8$ section with the same minimum, or $s(\Lambda) \le 99$, with equality only on the perfect lattice denoted by L_{99} in* [K-M-S].

It is generally believed[1] that the maximal kissing numbers up to dimension 25 are attained on laminated lattices, except $s(\Lambda_{12}^{\max}) = 324 < s(K_{12}) = 378$ and $s(\Lambda_{13}^{\max}) = 453 < s(K_{13}) = 459$.[2]

Indeed, in dimension 10, the largest known values of s is $s(\Lambda_{10}) = 168$. For the Barnes lattice $\langle \mathbb{E}_8, \mathbb{D}_{10} \rangle$ one has $s = 154$, and among perfect lattices contained in the Leech lattice Λ_{24}, one finds the values $s = 138$ (K_{10}) and $s = 135$ (K'_{10}); see the catalogue of lattices on my home page.

Question 2.3. *Do there exist other perfect, 10-dimensional lattices with $s \ge 135$?*

We have only considered the kissing number problem for lattices. For the general kissing number problem, the maximum values of s is known only in

[1] and proved for $n \le 8$ and $n = 24$

[2] Note that $s(\Lambda_{11}^{\max}) = 219 > s(K_{11}) = 216$ $(= s(\Lambda_{11}^{\min}))$, though $\gamma(K_{11})$ is (slightly) larger than $\gamma(\Lambda_{11}^{\max}) = \gamma(\Lambda_{11}^{\min})$

dimensions 2, 3, 4, 8 and 24. For $n = 2, 8, 24$, there is a unique solution, namely the configuration of minimal vectors of the lattices $\mathbb{A}_2$, $\mathbb{E}_8$ and Λ_{24} (the Leech lattice), respectively. The highest known values are obtained with sphere packings (most of the time, lattice packings), but uniqueness is often expected to fail, as in dimension 3, where the icosahedral configuration produces 12 spheres, as $S(\mathbb{A}_3)$. In dimension 4 it is not known whether configurations with $s = 24$ which are not similar to $S(\mathbb{D}_4)$ exist.

2.2. **Integral lattices having a small minimum.** The kissing number problem may be posed for integral primitive lattices having a given minimum m in any dimension $n \geq 2$. The case when $m = 1$ is trivial ($s_{\max} = n$, attained uniquely on $\mathbb{Z}^n$) and for $m = 2$, the classification of root systems provides a complete description (in particular, for $n \geq 17$, one has $s_{\max} = n(n-1)$, attained uniquely on the $\mathbb{D}_n$ lattice). No general result is known for $m \geq 3$. The odd minima, which scarcely occur among perfect lattices, are especially fascinating.

In [Mar-V2], lattices of minimum 3 having a somewhat large kissing number are classified up to dimension 9. Also, $s_{\max}$ is determined for $n \leq 7$ and all odd m, and for $n = 16, 22, 23$ and $m = 3$.

Problem 2.4. *To push further the results of [Mar-V2] for minimum $m = 3$. Does $s_{\max} = 40,\ 52,\ 68$, for $n = 10,\ 11,\ 12$, respectively? Also, for $n = 16$ and $m = 3$, where $s_{\max} = 256$ is attained only on the strongly perfect lattice O_{16}, is it true that $s \leq 190$ if Λ is not isometric to O_{16}? What about dimensions 22 and 23, for which $s_{\max}$ is again attained only on the strongly perfect lattices* O_{22} and O_{23} *?*
[For the notion of a *strongly perfect lattice*, see Subsection 4.7.]

2.3. **Perfect integral lattices having an odd minimum.** We scale perfect lattices to the minimum m which makes them integral and primitive (perfect lattices are rational), and again focus on those for which m is odd. Note that such a lattice has no $\mathbb{A}_2$-section with the same minimum. For $n \leq 7$, m is even except for $\mathbb{Z}$ ($m = 1$) and $P_7^2 \sim \mathbb{E}_7^*$ ($m = 3$). In [Wat7], Watson considered the more general question of classifying lattices (with $n \geq 2$) having no hexagonal section with the same minimum and such that $s \geq \frac{n(n+1)}{2}$ (the "perfection bound"), and proved that there exists a unique such lattice in the range $[2, 7]$, for which $m = 3$ and $s = \frac{n(n+1)}{2} (= 28)$ (actually, a scaled copy of $\mathbb{E}_7^*$).

In [B-M7], A.-M. Bergé and myself constructed perfect lattices with odd m in all dimensions $n \geq 10$, which left open only the cases $n = 8$ and $n = 9$. We knew that for $n = 8$, the 10916 perfect lattices quoted in [M] all have a hexagonal section with the same minimum, so that the proof in [D-S-V] that the list above is indeed complete settled the case of dimension 8.

Moreover, these authors produced for $n = 9$ a list of more than 500 000 perfect lattices, and Cordian Riener[3] has checked that no lattice of this list has an odd minimum.

Conjecture 2.5. *Every perfect,* 9*-dimensional lattice has a hexagonal section having the same minimum.*

A solution to the following problem could prove useful to handle Conjecture 2.5.

Problem 2.6. *Extend Watson's methods of [Wat7] to prove* a priori *that every perfect,* 8*-dimensional lattice has a hexagonal section having the same minimum.*

3. Minimal Classes

Given E of dimension $n \geq 2$, we consider on the set of lattices in E the equivalence relation

$$\Lambda \sim \Lambda' \iff \exists u \in \mathrm{GL}(E) \mid u(\Lambda) = \Lambda' \text{ and } u(S(\Lambda))) = S(\Lambda')\,;$$

classes for this relation are called *minimal classes*.
We also define *dual-minimal classes* by adjoining the third condition

$$u(S(\Lambda^*)) = S(\Lambda'^*)\,.$$

A detailed study of these two notions can be read in [M], Chapter 9. In particular, it is proved that there are only finitely many minimal classes in each given dimension, and moreover that every minimal class is the (disjoint) union of finitely many many dual-minimal classes, so that there also are only finitely many dual-minimal classes.

An important invariant of a minimal class $\mathcal{C}$ is the common perfection rank $r \in [1, \frac{n(n+1)}{2}]$ of the lattices belonging to $\mathcal{C}$, and we indeed have $r \geq n$ since we restrict ourselves to well rounded lattices.

The set of (positive, definite, having a prescribed minimum) quadratic forms carries a canonical structure of *cell complex*, the cells of which up to equivalence are in one-to-one correspondence with minimal classes; this complex plays a crucial rôle in the calculation of the cohomology of $\mathrm{SL}_n(\mathbb{Z})$ and of higher K-groups; see [E-G-S2]. In this correspondence, the dimension of a cell is the co-rank of the corresponding minimal class. I do not know any cohomological interpretation of dual-minimal classes.

[3] e-mail, March 9th, 2006

3.1. **Minimal classes.** The classification for $n \le 4$ (Štogrin, Bergé-Martinet) can be read in [M]; for $n = 5$ (Batut) and $n = 6, 7$ (Elbaz-Vincent, Gangl, Soulé), see [Bt] and [E-G-S2]. The number of cells is a quickly increasing (doubly exponential?) function of the dimension, from 18 for $n = 4$ to more than 10 000 000 for $n = 7$, so that a classification in larger dimensions is not feasible.

However a look at dimensions $5, 6, 7$ shows that classes of a given perfection rank r are numerous when the value of r is far from its minimal and maximal values (n and $\frac{n(n+1)}{2}$). Indeed classes of co-rank 0 are similarity classes of perfect lattices, thus correspond to perfect forms, and classes of co-rank 1 correspond to edges of the Voronoi graph. Thus even for $n = 8$ where this graph has more than eighty thousand edges, it is possible to list classes of small co-rank.

In the other direction, an explicit determination of classes with small $r - n$ (or small $s - n$, this essentially amounts to the same) is possible up to $n = 9$, using Watson's index theory; see Section 6 below. After Elbaz-Vincent, classification of minimal classes having a small kissing number could have applications to the cohomology of $\mathrm{SL}_n(\mathbb{Z})$. For this reason A.-M. Bergé and myself made some explorations in 2006, on which I intend to return. This is the origin of the following problem [4].

Problem 3.1. *List all minimal classes for* $n \le 9$ *and* $s = n, n+1, n+2$*, maybe also* $s = n + 3$.

3.2. **Dual-minimal classes.** In dimension 2, every lattice is similar to its dual by a similarity of symplectic type, of angle $\frac{\pi}{2}$, so that dual-minimal classes coincide with ordinary minimal classes, classified by the value of s (1, 2, or 3).

The classification for $n = 3$ was done by A.-M. Bergé in her 1995 paper [Ber1], in full generality. (In contrast with ordinary classes, the classification for non-well-rounded classes does not trivially results from classifications in lower dimensions.) In [M], Section 9.2, I only gave the proofs for well-rounded classes. Here I restrict myself to pairs $(\mathcal{C}, \mathcal{C}^*)$ with well-rounded $\mathcal{C}$, which suffices for the applications that we shall consider in the next sections; see in particular Subsections 4.3 and 5.3. Note that, since one may exchange a lattice with its dual, it suffices to consider minimal classes with $s \ge s^*$.

Problem 3.2.
(1) *Classify dual-minimal classes* $(\mathcal{C}, \mathcal{C}^*)$ *of dimension* 4 *such that* $\mathcal{C}$ *is well-rounded.*
(2) *Classify dual-minimal classes* $(\mathcal{C}, \mathcal{C}^*)$ *of dimension* 5 *such that* $\mathcal{C}$ *and* $\mathcal{C}^*$ *are well-rounded and* $s(\mathcal{C}) \ge 8$.

[4] on which there seems to be some work in progress around Dutour-Sikirić

[The two parts of the problem above are related to Questions 5.3 and 4.6 below, respectively.]

Extensions to dimensions 6 and 7 of Problem 3.2 look intractable. However non-classical dual-extreme lattices in these dimensions are likely to exist, and it would be interesting to construct new examples. (I remember having constructed with A.-M. Bergé a 7-dimensional analogue of her 5-dimensional example of [Ber4], using the regular representation of the cyclic group of order 7.)

4. Extremality Properties

The problems we are concerned with in this section belong to the following type: given a parametric family $\mathcal{F}$ of n-dimensional lattices, find the local maxima of density on $\mathcal{F}$ (*extreme $\mathcal{F}$-lattices*) and the absolute maxima (*absolutely extreme* or *critical $\mathcal{F}$-lattices*). The family $\mathcal{F}$ may consist of all lattices, or of *isodual* lattices (lattices isometric to their dual; maybe only up to scale), or more specifically *orthogonal* (resp. *symplectic* lattices), those for which the *isoduality* $\sigma : \Lambda \to \Lambda^*$ satisfies $\sigma^2 = \mathrm{Id}$, $\sigma \neq \pm\,\mathrm{Id}$ (resp. $\sigma^2 = -\,\mathrm{Id}$). Also the classification of local maxima of the *Bergé-Martinet invariant* $\gamma'(\Lambda) = (\gamma(\Lambda)\cdot\gamma(\Lambda^*))^{1/2}$ (equal to the usual Hermite invariant for isodual lattices) belongs to this type of problems; see [M], Section 10.5. The *Bergé-Martinet constant* is $\gamma'_n = \sup_{\dim\Lambda=n}\gamma'(\Lambda)$.

[We shall often consider ${\gamma'}^2$, which takes rational values on rational lattices (i.e., proportional to integral lattices), rather than γ' itself. However we cannot exclude that some γ'_n be attained on irrational lattices — it is only known that such lattices are *algebraic*, i.e., proportional to lattices having Gram matrices with coefficients in a number field.]

Scale an algebraic lattice Λ to a rational minimum m, say, $m = 1$. Since the base changes are defined over $\mathbb{Z}$, the number field defined by the coefficients of a Gram matrix for Λ does not depend on the chosen basis. Hence an algebraic lattice has a well-defined *field of definition*.

4.1. **Critical lattices.** Critical lattices (for the family of all lattices) are known in dimensions 1 to 8 (this dates back to Blichfeldt's 1935 paper [Bl2]) and in dimension 24, thanks to the recent work [Cn-Km1] of Cohn and Kumar. To try to fill the gap is a natural problem. The first case is dimension 9. It is known — this is what Chaundy proved in 1946 — that the laminated lattice Λ_9 is optimal among those which have an E_8 cross-section with the same minimum. Due to various recent progress in mathematics and in computer science, the following problem, that we quote for the sake of completeness, despite its lack of originality, is perhaps no longer completely intractable. The ideas described in Subsection 6.3 could be helpful.

Problem 4.1. *Prove that the laminated lattice* Λ_9 *is the only (up to similarity)* 9*-dimensional critical lattice.*
[Besides their work [D-S-V] in dimension 8, Dutour, Schürmann and Vallentin also ran the Voronoi algorithm for $n = 9$ during several months, producing more that 500 000 perfect lattices. A few years computation could well solve the question, but this is not what is really wanted!]

When γ_n is not known, the best known upper bounds for γ_n come from upper bounds of the density of *any* sphere packing.
[At the time I am writing this note,[5] the best estimates in the range $[4, 36]$ are those proved by Cohn and Elkies in [Cn-El].]

Problem 4.2. *Find specific sharper bounds for lattice packings. In particular, improve on the Cohn-Elkies bound for dimension* 9*.*

One expects the maxima of γ_n to be attained on the laminated lattices up to $n = 25$, except for $n = 11, 12, 13$ where the candidates are the K_n. For $n = 26$, there exist laminated and non-laminated lattices with the same Hermite invariant, and for $n = 27, 28, 29$, Bacher ([Bc1]) has found examples showing that the laminated lattices are no longer optimal.

Question 4.3. *Do there exist for* $n \in [26, 31]$ *better examples than Bacher's or those obtained as cross-sections of the known extremal symplectic lattices of dimension* 32 *?* (See [M] and [M'], Chapter 14, and Subsection 4.2 below.)

4.2. **The Bergé-Martinet constant.** One has $\gamma'_n \le \gamma_n$, and equality holds if and only if the dual of some critical lattice in dimension n is also critical. This applies to dimensions $n = 2, 4, 8, 24$, and conjecturally to $n = 12$ and 16, since (putative) critical lattices in these dimensions are isodual (and more precisely, symplectic).

Problem 4.4. *Find bounds for* γ'_n *which are sharper than those provided by the inequality* $\gamma'_n \le \gamma_n$*; prove an inequality of the form* $\gamma'_9 \le a < 2$*.*
[Strict inequalities $\gamma'_n < \gamma_n$ have been proved for $n = 3$, 5, 6 and 7; see below, and the *strict* inequality $\gamma'_9 < 2$, which implies $\gamma'_9 < \gamma_9$, can be proved using Theorem 2.8.7 (2) of [M] and the now known exact value of γ'_5, observing that equality may not hold in Theorem 2.8.7 (2). In general,

[5] (April 5th, 2016.) Optimal bounds have been found for $n = 8$ and $n = 24$; see [Viaz] and [C-K-V-al]

it would be interesting to find analogues of the techniques (going back to Minkowski) making use of measure theory. One could perhaps make use of the interpretation of $\gamma'(\Lambda)$ in the setting of Lie groups (see [M], Chapter 10).]

Problem 4.4 is certainly difficult. In the other direction, it is also interesting to construct examples in low dimensions for which the duals to the putative critical lattices have a bad Hermite invariant. The results displayed in [M], Chapter 14; see also [M'] (which owe much to Conway-Sloane's paper [C-S9]) could perhaps be improved, and it would also be interesting to go beyond dimension 24, up to dimension 32, where nice, symplectic lattices, with $\gamma^2 = {\gamma'}^2 = 18$ are known.

Problem 4.5. *Construct lattices having a high Bergé-Martinet invariant in dimensions* $9-23$ *and* $25-31$*. In particular, if* $n = 9$*, are there lattices having* ${\gamma'}^2 > \frac{16}{5}$ *? Find an explicit* a *such that* $\gamma'_9 \leq a < 2$*.*
[The largest known value for $n = 9$ is attained on (A_9^2, A_9^5) and on another pair (L, L^*) found by Baril in [Bari]; see [M'], Section 3.8 C.]

A neighbour problem is that of the Rankin constants and their dual forms $\gamma_{n,k}$ and $\gamma'_{n,k}$, that need be considered only for $k \leq \frac{n}{2}$ by symmetry and for $k > 1$ since $\gamma_{n,1} = \gamma_n$ and $\gamma'_{n,1} = \gamma_n$. After work of Poor and Yuen ([P-Y4]) and Watanabe et al. ([S-Wt-O]) these constants are known in low dimensions $n \leq 8$ except if $(n, k) = (5, 2)$, $(6, 3)$, $(7, 2)$ and $(7, 3)$, where they are expected to be attained uniquely on the root lattices $\mathbb{D}_5$, $\mathbb{E}_6$, $\mathbb{E}_7$ (and their duals for $\gamma'_{n,k}$).

4.3. **Dual-extreme lattices. I: low dimensions.** These are the lattices (always considered up to similarity) on which γ' attains a local maximum. They are called *dual-critical* if this is an absolute maximum. These notions were introduced in the 1989 paper [B-M1], in which dual-extreme lattices in dimensions $n \leq 4$ were classified (these are $\mathbb{Z}$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_3{}^*$, $\mathbb{D}_4$, $\mathbb{A}_4$, $\mathbb{A}_4{}^*$, the extreme lattices and their duals). The constant γ'_n and the corresponding dual-critical lattices are also known in dimensions 8 and 24 (see above), and thanks to recent work by Poor and Yuen ([P-Y4]), in dimensions $5, 6, 7$ (these are the extreme lattices $P_5^1 \simeq \mathbb{D}_5$, Coxeter's $P_5^2 \sim \mathbb{A}_5^3$, $P_6^1 \simeq \mathbb{E}_6$, $P_7^1 \simeq \mathbb{E}_7$ and their four duals).

In dimension 5, there are 8 known dual-extreme lattices: the three extreme lattices, their three duals, and an irrational pair (L, L^*) defined on $\mathbb{Q}(\sqrt{13})$, found by A.-M. Bergé ([Ber4]).

Question 4.6. *Are the eight lattices above the only dual-extreme,* 5*-dimensional lattices?*
[Since we must have $s(\Lambda) + s(\Lambda)^* \geq \frac{n(n+1)}{2} + 1 = 16$ (see next subsection) it

suffices to consider lattices with $s \geq 8$. Unpublished (and lost) calculations done by A.-M. Bergé have probably shown that the only examples for $s \geq 11$ are the three extreme lattices. (She actually solved problem 3.2 for $s \geq 11$.) There remains to consider the minimal classes having $s = 10$ (where three dual-extreme are known), 9 and 8, a more and more complicated task, since the number of parameters is then $15 - r = 15 - s$. We shall return later to this classification problem in relation with (dual-)minimal classes.]

4.4. **Dual-extreme lattices. II: theoretical questions.** For a dual-extreme lattice Λ, we have $s(\Lambda) + s(\Lambda^*) \geq \frac{n(n+1)}{2} + 1$. In [B-M6], dual-extreme lattices meeting this bound were constructed in all even dimensions $n \geq 8$, whereas for $n = 2, 3, 4$, the lower bound of $s + s^*$ is equal to $\frac{n(n+1)}{2} + (n+1)$ (= 6, 10, 15, respectively).

Problem 4.7. *Do dual-extreme lattices with $s(\Lambda) + s(\Lambda^*) = \frac{n(n+1)}{2} + 1$ exist in all large enough odd dimensions? What is the precise lower bound of $s + s^*$ (among dual-extreme lattices) for $n = 5, 6, 7$?*

4.5. **HKZ reduction.** For the notation I refer to [B-M1] and [M], Section 2.9. I consider the *external* (or *outer*) coefficients A_i which occur in decompositions into squares of positive, definite quadratic forms, and in particular $\inf A_1/A_i$ for $1 \leq i \leq n$ (= $\inf A_{k+1}/A_{k+i}$ for any $k \in [0, n-i]$), the square of $\frac{1}{\gamma''}$ in the notation of [B-M1]), and the upper bound γ''_n of γ'' on the set of n-dimensional lattices. Hermite found the exact bound $A_1/A_2 \geq \frac{3}{4}$, and used it to deduce the lower bound $A_1/A_n \geq (\frac{3}{4})^{n-1}$, thus $A_1/A_3 \geq \frac{9}{16}$. Korkine and Zolotareff proved in [K-Z2] the subtle lower bound $A_1/A_3 \geq \frac{2}{3}$, which may be attained only on lattices for which $\frac{3}{4} \leq A_2/A_3 \leq \frac{8}{9}$, and is indeed attained exactly on the similarity classes of $\mathbb{A}_3$ (with $A_2/A_3 = \frac{3}{4}$) and $\mathbb{A}_3^*$ (with $A_2/A_3 = \frac{8}{9}$). This then shows the lower bound $A_1/A_4 \geq \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$, which is indeed exact. In [Bl2] Blichfeld gives an example which shows that "the expected bound $A_1/A_5 \geq \frac{1}{2}$" is not correct, an example which suggests the conjectural exact bound $A_1/A_5 \geq \frac{15}{32}$. This conjecture has been "almost" proved by Pendavingh and van Zam ([Pe-vZ], 2007), who showed that we have $A_1/A_5 \geq \frac{15}{32} - 2 \cdot 10^{-5}$.

Problem 4.8. *Prove that the lower bound $\frac{15}{32}$ is exact and characterize the forms which on which equality holds. Find good lower bounds for A_1/A_n, $n = 6$ to 9.*

The invariant γ'' is not a continuous function on the set of lattices. This lead Gindraux ([Gi']) to introduce two invariants γ''^{+} and γ''^{-}. It would be interesting to study more closely these invariants.

4.6. **Isodual lattices.** In his paper [Bav5], Bavard obtained important classification results for isodual lattices Λ of orthogonal type and signature $(n-1,1)$ ("Lorenzian orthogonal lattices"), and he recently extended these results to other signatures in [Bav6], obtaining in particular almost complete results in dimensions $n \leq 5$.
[In the notation above, the signature (p,q) is that of the quadratic form $x \cdot \sigma x$; we may assume that $p \geq q$, and disregard the case when $q = 0$, where Λ is unimodular; moreover, there is an extra invariant (the *parity*) when $p \equiv q \mod 8$.]
Note that the weaker notion of a *normal lattice* ([M], Def. 11.5.4 and Section 11.6) could be useful when dealing with isodual lattices.

An important, still opened question is:

Problem 4.9. *Classify extreme-symplectic* 6*-dimensional lattices; in particular, prove that the densest lattice is the lattice over* $\mathbb{Q}(\sqrt{3})$ *constructed in [C-S9] with glue vectors and viewed in [M] as* the *isodual lattice lying on the Voronoi path* $\mathbb{E}_6 — \mathbb{E}_6^*$.

4.7. **Strongly eutactic and strongly perfect lattices.** These notions were introduced by Boris Venkov [6] in [Ven3]. A lattice Λ is *strongly eutactic* (resp. *strongly perfect*) if its set of minimal vectors is a 3- (resp. a 5-spherical design). Strongly eutactic lattices are simply those which have a eutaxy relation with equal coefficients; see Section 5 below. We now restrict ourselves to strong perfection. It is proved in [Ven3] that strongly perfect lattices are indeed extreme, but for $n \geq 3$, strong perfection is a much more restrictive notion than the mere extremality. The classification of strongly perfect lattices is known up to $n = 12$ ([Ven3], [Ne-V1], [Ne-V2]). Work in progress in Aachen in Gabriele Nebe's team is expected to extend the known classification results, maybe under additional hypotheses; see in particular [Ne-V6] and [Nos2].

Also, strongly perfect *integral* lattices with minimum $m \leq 3$ have been classified in [Ven3].

Problem 4.10.
(1) *Are there strongly perfect lattices in dimensions* $n \leq 23$ *other than those (found by Batut and Venkov) which are listed in [Ven3], Section 19?* [7]
(2) *Describe integral strongly perfect lattices with minimum* 4 *or* 5.

---

[6] Boris B. Venkov passed away on November 10th, 2011

[7] (August 25th, 2018.) Some extist in dimension 16; see [Hu-Ne].

[(1) & (2) For 7-designs, see [Ne-V5] and [Mar7].
(2) Their dimensions are bounded from above, but I do not ask for a classification because unimodular lattices of minimum 4 and dimension 32 are strongly perfect but have not been classified.]

As for designs of higher level I state here as a conjecture the three questions I asked at the end of [Mar7].

Conjecture 4.11. *The highest level of a spherical design afforded by the minimal vectors of a lattice* Λ *of dimension* $n \geq 2$ *is equal to* 1, 3, 5, 7 *or* 11*, and the level is equal to* 11 *if and only if* Λ *is a unimodular lattice of dimension* $n \equiv 0 \mod 24$.
[Note. Level 7 is known to occur only for even unimodular lattices of dimension $n \equiv 8 \mod 24$, for even 2-modular lattices of dimension $n \equiv 0 \mod 16$, and for the (odd) *shorter Leech lattice* O_{23}.]

There is a curious example in dimension 21, namely the lattice K'_{21}: it is strongly perfect, but its dual is not (it is only strongly eutactic). No other example is known, but one cannot reasonably expect K'_{21} to be the only one.
[As was pointed out to me by Venkov, most of the strongly perfect perfect lattices have been constructed using techniques (invariants of groups, modular forms) which apply to all levels of a lattice and of its dual; this is perhaps the main reason for which K'_{21} appears to be an isolated example.]

The question I want now to discuss concerns the construction of infinite series of strongly perfect lattices. For the while the only known infinite series is that of the Barnes-Wall lattices BW_n for $n = 2^k \geq 4$ $(\mathbb{D}_4, \mathbb{E}_8, \Lambda_{16}, \dots)$, which are even 7-designs for $n \geq 8$. I explain below a construction of these lattices which could be applied in other situations.

Let L be an integral lattice, of dimension n, equipped with a "symplectic" automorphism σ (i.e., we have $\sigma^2 = -\mathrm{Id}$; in other words, L is a Hermitian $\mathbb{Z}[i]$-lattice, a *Gaussian lattice*). View $L \times L$ as the orthogonal sum $L \perp L$ inside $E \times E$, then consider the sublattice of $L \perp L$ defined by the congruence $x \equiv y \mod (1+\sigma)L$, and go on. One can rescale every two steps the lattices to half their norms, obtaining this way a series $L_0 \subset L_1 \subset L_2 \subset \dots$ of lattices which are even from L_2 or L_3 on.

In this sequence the annihilators of L_i^*/L_i are alternatively multiplied and divided by a factor 2. Thus if L_0 is unimodular, the L_i are alternatively 2- and 1-modular. This is the case of the $L_i = \mathrm{BW}_{2^{i+1}}$, starting with $\mathbb{Z} \times \mathbb{Z}$ and $\sigma(x, y) = (-y, x)$.

Question 4.12. *Can one construct other infinite series of strongly perfect lattices by the method above applied with other lattices* L *than* $\mathbb{Z}$ *?*

A good candidate for L could be the Leech lattice Λ_{24}, with its unique conjugacy class of symplectic automorphisms. The first step is a 2-modular lattice of dimension 48 (Quebbemann), indeed the laminated lattice of Conway-Sloane's "principal series". One could also look at 32-dimensional lattices (not BW_{32} !) — many of them may be expected to be of symplectic type.

The direct study of minimal vectors (as in [Ven3] for the Barnes-Wall series) is certainly difficult; maybe group theory can provide examples to which the theory of invariants apply (with the Leech lattice?), as in [Bac3] for the Barnes-Wall series.

One more remark on the construction above: it is easy to list all extremal (in the sense of modular forms) 1- or 2-modular lattice of dimension n which can be constructed by the method above from lattices of dimension $\frac{n}{2}$. This stops at $n = 64$, where unimodular lattices of minimum 6 can be constructed from the four known extremal 2-modular lattices of dimension 32. But I do not know how to prove that we indeed obtain pairwise non-isometric lattices.

Finally I mention a possible connection of the theory of strong eutaxy with Watson's index theory of Section 6.

Conjecture 4.13. *In any infinite series of strongly perfect lattices, the maximal index tends to infinity with the dimension.*
(Stronger form: even the maximal annihilator of a quotient Λ/Λ' where Λ' is generated by minimal vectors of Λ tends to ∞.)

5. Eutaxy and Related Properties

5.1. **Weak eutaxy.** We say that a lattice is *weakly eutactic* if there exists a relation of the form $\mathrm{Id} = \sum_{x\in S(\Lambda)/\pm} a_x\, p_x$ with real *eutaxy coefficients* a_x; expressed using a Gram matrix $A = \mathrm{Gram}(\mathcal{B})$ for some basis $\mathcal{B}$ for Λ, this reads $A^{-1} = \sum_{x\in S(A)/\pm} a'_x\,(x^t x)$. If it is possible to find non-negative (resp. strictly positive) eutaxy coefficients, we say that Λ is *semi-eutactic* (resp. *eutactic*). It results from work of A.-M. Bergé and myself that there are only finitely many n-dimensional weakly eutactic lattices (at most one per minimal class), and that these lattices are algebraic.

Problem 5.1. (1) *Use the classification of cells obtained by Elbaz-Vincent, Gangl and Soulé in [E-G-S2] to classify weakly eutactic lattices in dimensions* 6 *and* 7*.*
(2) *Is there an interpretation in terms of lattices of the discriminant and the signature of the field of definition?*

(3) *Is every number field with at least one real place the field of definition of some (weakly) eutactic lattice?*

[(1) After Batut, one can use the *gradient algorithm*; see [Bt], which proved efficient in dimension 5.
(2) Up to dimension 5, eutactic lattices are defined over totally real fields. Batut (unpublished, and probably lost) has classified the eutactic lattices belonging to a minimal class $c \prec cl(A_6)$. He found this way an example defined over a cubic field with mixed signature.
(3) is part of Question 9.7.4 of [M], where variants of eutaxy to be discussed below are also considered.]

5.2. **Strong eutaxy.** Since the notion of strong eutaxy is less restrictive than that of strong perfection, classifying strongly eutactic lattices in a given dimension will be even more difficult than classifying strongly perfect lattices in this dimension. This looks hopeless in dimensions $n \geq 8$. For $n \leq 6$, this is a byproduct of the classification of minimal classes (see [M] for $n \leq 4$; [Bt] for $n = 5$; [E-G-S2] for $n = 6$); the results can be read on my home page. Dimension 7 could be obtained using the data of [E-G-S2]. Indeed, a minimal class $\mathcal{C}$ is characterized by its *barycenter matrix* $S \cdot {}^t S$ where S is the matrix of components of $S(\Lambda)$ of some $\Lambda \in \mathcal{C}$ on a basis for Λ. (In other words, $\mathcal{C}$ is characterized by the isometry class of a lattice with Gram matrix $S \cdot {}^t S$.) Then $(S \cdot {}^t S)^{-1}$ is a Gram matrix for the strongly eutactic lattice in $\mathcal{C}$, if any.

The only general classification results that can be reasonably expected concern lattices with "small" $s - n$.

Problem 5.2. *Classify strongly eutactic lattices which are generated by their minimal vectors and have $s = n + 1$, $s = n + 2$ or $s = n + 3$.*
[We find only $\mathbb{Z}^n$ for $s = n$ (weak eutaxy suffices), and probably only $\mathbb{A}_n^*$ for $s = n + 1$.]

5.3. **Dual-eutaxy.** In [B-M1]; see [M], Section 3.8, dual-extreme lattices were characterized à la Voronoi by properties of (1) dual-perfection and (2) dual-eutaxy, namely (1): the p_x, $x \in S \cup S^*$ span $\mathrm{End}^s(E)$, and (2): there exists a relation of the form

$$\textstyle\sum_{x \in S(\Lambda)/\pm} a_x p_x = \sum_{x \in S(\Lambda^*)/\pm} a'_x p_x$$

with strictly positive coefficients a_x, a'_x. Under these conditions, both S and S^* span E, so that we may define *dual-perfection* and *dual-eutaxy* by adding to conditions (1) and (2) above the fact that *each set S and S^* spans E*. Thanks to these complementary conditions, it can be proved ([M], Th. 9.6.10), that there are only finitely many dual-eutactic lattices (up to similarity), and that these lattices are algebraic.

If $n = 2$, since all lattices are isodual, dual-eutaxy is equivalent to eutaxy, and holds uniquely for $\mathbb{Z}^2$ and $\mathbb{A}_2$.

If $n = 3$, dual-eutactic lattices are $\mathbb{Z}^3$ ($s = s^* = 3$), $\mathbb{A}_3$ ($s = 6, s^* = 4$) and $\mathbb{A}_3^*$, and the "*ccc*-lattice" ($s = s^* = 4$), defined over $\mathbb{Q}(\sqrt{2})$; see [M], exercise 9.6.2 and Proposition 11.6.1.

Problem 5.3. *Use Problem 3.2, (1), to list all* 4*-dimensional dual-eutactic lattices. In particular, is* $\mathbb{Z}^4$ *the unique such lattice with* $s = s^* = 4$ *? More generally, what about* $s = s^* = n$ *?*

6. Watson's Index Theory

The aim of this theory, initiated by Watson in [Wat4], is to classify pairs (Λ, Λ') of a well-rounded lattice Λ and a (well-rounded) sublattice Λ' of Λ having a basis made of minimal vectors of Λ.

Let d be the annihilator of Λ/Λ'. We have $d \le [\Lambda : \Lambda']$, and $[\Lambda : \Lambda'] \le \gamma_n^{n/2}$ (use Hadamard inequality), hence $d \le \gamma_n^{n/2}$. We may express Λ on a basis of minimal vectors $(e_1, \dots, e_n)$ for Λ' as $\Lambda = \langle \Lambda', x_1, \dots, x_k \rangle$ where the x_i are of the form $x_i = \dfrac{a_1^{(i)} e_1 + \cdots + a_n^{(i)} e_n}{d}$. The numerators of the x_i define a linear code $\mathcal{C}$ over $\mathbb{Z}/d\mathbb{Z}$. The precise question is now to list the codes which may occur in a given dimension. (Warning: for $d > 2$, it may happen that a code does not correspond to a pair (Λ, Λ'). For instance, for $n = 9$, there exists a ternary code of dimension 3, but the maximal possible index is $16 < 3^3$.)

The results are known up to dimension $n = 9$:[8] for $n \le 8$, they can be read in [Mar6], which extends previous work by Watson, Ryshkov and Zahareva. The case of dimension 9 has been solved recently by Keller, Schürmann and myself ([K-M-S]). In particular, we obtain from the results above the list of all possible structures of Λ/Λ' as an Abelian group, and the *maximal index* $\imath_n = \max_{\Lambda'} [\Lambda : \Lambda']$ for $n \le 9$.

One more notation for further use: reordering the basis (e_i) and reducing the $a_j^{(i)}$ modulo d if need be, we may assume that for some $\ell \le n$, we have $1 \le |a_j^{(i)}| \le \frac{d}{2}$ if $j \le \ell$ and $a_j^{(i)} = 0$ if $j > \ell$. We call ℓ the *length of* (Λ, Λ'). Thus if $d = 2$ (resp. $d = 3$), we may assume that $a_j^{(i)} \in \{0, 1\}$ (resp. $a_j^{(i)} \in \{0, \pm 1\}$). Moreover, negating some e_i if need be, we may assume whatever d that $a_j^{(1)} \ge 0$.

[8](August 25th, 2018.) In a mail from December, 2016, M. Dutour-Sikirić informed me that he solved the case $n = 10$. In work in progress he also considers dimensions 11 and 12.

6.1. **Large indices.** Whereas dimensions $n \leq 8$ could be essentially dealt with by hand, dimension 9 could not have been handled without heavy computer-aided proofs. One of the problems is that on the one hand, the discrepancy between known upper bounds for $\gamma_n^{n/2}$ and conjectural values quickly increases with n, and on the other hand, the actual value of $\imath_n$ is often smaller than $\gamma_n^{n/2}$: equality $\imath_n = \gamma_n^{n/2}$ holds for $n = 4$, 7, 8, 24, conjecturally for $n = 16$ (with Λ_{16}), but is probably scarce (note that in these cases, the index is a power of 2); and moreover, large values of $\imath$ generally correspond to non-cyclic quotients Λ/Λ'. All this makes dimension 10 hardly feasible.

I have no general conjecture to set. For relatively small n, we observe that binary codes give for $n = 9$ to 13 examples of 2-elementary quotients of order 2^{n-5} (e.g., $\Lambda_n^{\max}$ for $n = 11, 12, 13$, $\langle\, \mathbb{E}_8, \mathbb{D}_{10}\,\rangle$ for $n = 10$).

Conjecture 6.1. *For* $n = 10, 11, 12, 13$*, one has* $\imath_n = 2^{n-5}$.

Question. For $n = 10$ and $\imath = 2^{n-5} = 32$, are the three types (2^5), $(4 \cdot 2^3)$ and $(4^2 \cdot 2)$ the only possible structures for Λ/Λ' ?

6.2. **Connection with perfection: small indices.** We first set the following conjecture, based on experimentation in low dimensions to be discussed below.

Conjecture 6.2. *For any integer* $k \geq 2$*, there exists* n_0 *such that every perfect lattice* generated by its minimal vectors *of dimension* $n \geq n_0$ *either is similar to the root lattice* $\mathbb{A}_n$ *(of index* 1*) or has index* $\imath \geq k$. [The condition *generated by its minimal vectors* is necessary; see below.]

I am not able to make a precise general conjecture giving an estimate of n_0 as a function of k. This looks however possible for small k.

By a theorem of Korkine and Zolotareff (see [M], Theorem 6.1.2), a perfect lattice of maximal index 1 is similar to the root lattice $\mathbb{A}_n$ (and the weaker condition " $s \geq \frac{n(n+1)}{2}$ " even suffices).

To discuss larger indices, we introduce the lattice Λ_0 generated by the set of minimal vectors of Λ, which necessarily contains all possible lattices Λ'.

When $\imath(\Lambda) = 2$, the set of indices of Λ is $\{2\}$ or $\{1, 2\}$ according to whether $\Lambda_0 = \Lambda'$ or $\Lambda_0 = \Lambda$. In the first case, Λ' is well-defined, and perfect (because $S(\Lambda_0) = S(\Lambda)$), hence similar to $\mathbb{A}_n$. As a consequence, Λ is of the form $\langle \mathbb{A}_m^2, \mathbb{A}_n \rangle$ for some odd integer $m \in [9, n]$ (see [M], Section 5.3 for the definition of the Coxeter lattices $\mathbb{A}_n^r$, $r \mid n+1$); there are $\lfloor \frac{n-7}{2} \rfloor$ such lattices, all perfect of index 2 but not generated by their minimal vectors.

In the second case, among perfect lattices with $n = 4, 5, 6, 7$, there are 1, 2, 2 and 1 lattices of index 2, and none exist for $n = 8$. Moreover, these are the less dense (or the two less dense) lattices after $\mathbb{A}_n$.

Similarly, there are only two 8-dimensional perfect lattices with maximal index 3.

These observations support the following conjecture:

Conjecture 6.3. *A perfect lattice which is* generated by its minimal vectors *and has maximal index $\imath \le 3$ is of dimension $n \le 8$ or is similar to $\mathbb{A}_n$.* [See [Ber6] for a partial result on index 2.]

A neighbour problem concerns the strict inclusions which may exist among perfect lattices having the same dimension. The only such pairs (Λ, Λ') in dimensions $n \le 8$ are are $(\mathbb{E}_7, \mathbb{A}_7)$, $(\mathbb{E}_8, \mathbb{A}_8)$, $(\mathbb{E}_8, \mathbb{D}_8)$.

Problem 6.4. *Classify all inclusions $\Lambda \supset_{\neq} \Lambda'$ between 9-dimensional perfect lattices.*

6.3. **Connection with perfection: index and density.** Recall that the perfect lattices are denoted up to $n = 7$ by P_n^i with decreasing density as i increases from 1 to a value i_n, equal to 33 for $n = 7$, and which would be equal to 10916 for $n = 8$. With respect to the density, Laïhem lattices — those which have a perfect hyperplane section with the same minimum — play a special rôle: among perfect, 8-dimensional lattices, the 8 densest lattices and the 23 less dense lattices are Laïhem lattices.

Problem 6.5. *Classify perfect, 9-dimensional lattices having a hyperplane section with the same minimum.*

A look at the long table of perfect lattices for $n = 8$ shows that most of the lattices have index in a middle range: 8072 lattices have index system $\mathcal{I} = \{1, 2, 3, 4, 2^2\}$, and the index system of 2837 out of the remaining 2844 ones is obtained from $\mathcal{I}$ by adjunction of $\{5\}$, $\{6\}$, or $\{5, 6\}$. In this respect a great deal of 9-dimensional perfect lattices is expected to have and index systems containing $\{1, ..., 7\}$, but I cannot forecast any precise result.

Clearly, a large index implies a high density. The problem is to know whether the converse holds *among perfect lattices* and to understand why the sets of indices are almost never highly lacunary (as $\mathbb{D}_n$, which has only the 2-elementary quotients of order 1, 2,. . . ,$2^{\lfloor (n-3)/2 \rfloor}$).

The case of small indices was considered in Subsection 6.2. In the other direction, we ask for good bounds of $\gamma(\Lambda)$ valid for perfect lattices having a not too large index. Actually, if $n = 7$, a direct proof of $\imath(\Lambda) \le 4 \Longrightarrow \gamma(\Lambda) < 2^{6/7} = 1.81...$ would suffice to prove that $\gamma_7 = 2^{6/7}$

and that this value is attained only on lattices similar to $\mathbb{E}_7$. Similarly (however, less important, since the transition from $n = 7$ to $n = 8$ can be easily done using the Mordell inequality), a proof of $\imath(\Lambda) \le 6 \Longrightarrow \gamma(\Lambda) < 2$ would solve the analogous problem for $n = 8$.

I end this section by quoting a conjecture of Coxeter.

Conjecture 6.6. (Coxeter.) *$\mathbb{A}_n$ is the less dense of the n-dimensional lattices.*

Formerly I had the opinion that it should be false. However a look at the tables of perfect lattices up to $n = 8$ shows that there is a gap of density between $\mathbb{A}_n$ and the other perfect lattices. Moreover taking into account the loose link which seems to exist among perfect lattices between maximal index and density together with Conjecture 6.3 above, I have changed my mind.

7. Lattices Modulo 2

This section refers to my papers [Mar8], *Reduction Modulo 2 and 3 of Euclidean Lattices*, and its complement [Mar10], except that I restrict here myself to modulus 2. In these papers I considered the question of finding representatives of small norm for the $2^n - 1$ non-zero classes modulo 2 of an n-dimensional lattice Λ. Let $m = \min \Lambda$. Given $x, y \in \Lambda$ with $y \equiv x \mod 2$, say, $y = x + 2z$, we have the identity

$$N(y) + N(x) = 2(N(x + z) + N(z))\,, \qquad (*_7)$$

from which we deduce the following three properties which hold when both x, y have norm at most $2m$:

1. (1) $N(y) = N(x)$.
2. (2) If $N(x) < 2m$, $y = \pm x$.
3. (3) If $N(x) = 2m$, such vectors y constitute an orthogonal frame S_x with $k(x) \le n$ pairs $\pm y$, and the vectors $\frac{y+z}{2}$, $y, z \in S_x$ constitute a root system of type $\boldsymbol{D}_k$ (scaled to norm m).

Set $k(x) = 1$ if $0 < N(x) < 2m$. We then have

$$\sum_{0<N(\pm x)\le 2m} \frac{1}{k(x)} \le 2^n - 1$$

and equality holds if and only if all classes of Λ modulo 2 contain a representative of norm $N \le 2m$. Also every $x \in \Lambda$ with $N(x) \le 2m$ is a minimal vector of some class modulo 2 of Λ.[9]

[9] by $(*_7)$, if Λ is integral, this holds for $N(x) \le 2m + 1$

This raises the following question: *when do all classes modulo* 2 *contain a vector of norm* $N \le 2m$ *?*

In [Mar8] such lattices are listed in dimensions 2 to 10, 12 (Coxeter-Todd's K_{12}) and 24 (Leech's Λ_{24}). Moreover in dimensions 2 to 6, for some of these lattices, the strict inequality $N < 2m$ suffices (which amounts to saying that these lattices have empty spheres of norm $2m$). This implies that this property then extends to a small enough neighbourhood, and shows that in dimension 2 to 6, the set of lattices for which $N \le 2m$ suffices contains an open subset in the set of n-dimensional lattices.

Question 7.1. *Do there exists lattices in dimensions* $n > 10$*, not similar to* K_{12} *nor to* Λ_{24}*, for which vectors of norm* $N \le 2m$ *suffice to represent all classes modulo* 2*?*

Question 7.2. *Do there exist lattices in dimensions* $n > 6$ *for which vectors of norm* $N < 2m$ *suffice to represent all classes modulo* 2*?*

My feeling is that the answers to the two questions above are negative. As a consequence it would be possible to classify lattices of dimension $n > 6$ having representatives modulo 2 of norm $\le 2m$.

Congruences modulo 2 between vectors of norm $> 2m$ and a given vector of smaller norm sometimes involve interesting configurations, in particular for pairs of norms of the form $(N, 4m - N)$. Also among (integral) odd lattices, one needs consider vectors of norm $2m + 1$ (if any) to obtain all classes modulo 2. In this case configurations $S(\mathbb{A}_\ell^*)$ play the rôle that payed $S(\mathbb{Z}^\ell)$ for vectors of norm $2m$.

8. Connection with Algebraic Number Theory

I just quote here a few questions related to algebraic structures on lattices. Well-known examples are Eisenstein, Gaussian or Hurwitz lattices. In full generality we wish to consider an order $\mathfrak{O}$ in some semi-simple algebra L, most of the time an algebra with involution, equipped with a *bilinear twisted trace form* $\mathrm{Tr}_{L/\mathbb{Q}}(\alpha x \overline{y})$ for some conveniently chosen $\alpha \in L$, and lattices Λ endowed with a structure of $\mathfrak{O}$-module for which the scalar product on $E := L \otimes_{\mathfrak{O}} \Lambda$ is obtained using the twisted trace form on L.

Important examples are provided by *G-lattices* where G is a finite group and L is a quotient of the group algebra $\mathbb{Q}[G]$. (In other words, E defines a representation of G over $\mathbb{Q}$.) Taking $G = \{1, \sigma, \sigma^2\}$ with $\sigma^3 = 1$, and assuming that σ is an orthogonal transformation of E with minimal polynomial $X^2 + X + 1$, we obtain *Eisenstein lattices*, since $\mathbb{Z}[G]/(1 + \sigma + \sigma^2)$ can be identified with the ring $\mathbb{Z}[\omega] \subset \mathbb{Q}(\omega) \simeq$

$\mathbb{Q}(\sqrt{-3})$, where ω is a cube root of unity and the scalar product comes from a Hermitian scalar product on E viewed as a vector space over $\mathbb{Q}(\sqrt{-3})$, the field of Eisenstein numbers. With $G = \{1, \sigma, \sigma^2, \sigma^3\}$, σ of minimal polynomial $X^2 + 1$, we obtain *Gaussian lattices*, and *Hurwitz lattices* which are modules over the order $\mathfrak{M}_2$ of *Hurwitz quaternions* inside the field of "usual" quaternions over $\mathbb{Q}$, with basis (in the usual notation) $1, i, j, \omega = \frac{-1+i+j+k}{2}$, where $i^2 = j^2 = -1$ and $ij = -ji = k$ are related to the double cover $\widehat{A}_4$ (or $\widetilde{A}_4$) of the alternating group on four letters; for a general discussion of Hermitian structures, see [M], Chapter 8.

Other interesting cases are cyclotomic structures (*cyclic G-lattices*) where $G = \langle\sigma\rangle$ is cyclic of some order $m \geq 3$ and the representation maps σ onto a a root of unity of order m in $\mathbb{Q}(\zeta_m)$, or quaternions with center $\mathbb{Q}$ ramified at 3 and ∞, where a maximal order $\mathfrak{M}_3$ affords a representation of the quaternion group H_{12}, or also quaternions with center $Q(\sqrt{2})$ or $Q(\sqrt{5})$, unramified off the two infinite places, which are connected with representations of the double covers $\widehat{S}_4$ of S_4 and $\widehat{A}_5$ (or $\widetilde{A}_5$) of A_5.

There is a Voronoi theory and a Voronoi algorithm for G-lattices which allows classifications; see [M], Chapters 11 and 12. Thanks to work of Sigrist, and more recently Schürmann, using the Voronoi algorithm, the classification of Eisenstein lattices is known up to dimension 10 (relative dimension 5) and that of Hurwitz lattices up to dimension 12. (Using this last result, Stephanie Vance could prove ([Van1]) that the density of Λ_{16} is the highest possible among Hurwitz lattices.)

It could be interesting to consider other structures than Eisenstein, Gauss and Hurwitz. To my knowledge no computations have been carried out for other groups in relative dimension > 1. Classifications for cyclic groups in relative dimension 1 have been obtained by Sigrist, e.g., groups of order 17 or 32 in dimension 16.

Independently of group theory, general classification problems, both in an algebraic or in an arithmetic setting, for forms $\mathrm{Tr}_{L/K}(\alpha xy)$, maybe twisted by a Galois action of order 2, have been considered by Eva Bayer and its Lausanne group. This can be viewed as a different approach of relative 1-dimensional problems.

A challenging problem is the possibility of constructing a given lattice as an ideal equipped with a (twisted) bilinear trace form. For instance such a construction of the lattice $\mathbb{E}_8$ has been obtained by Bayer-Fluckiger and Suarez in [Bay-S1] inside a totally real octic field L. Their construction makes use of the existence of a quartic field inside L.

Can one construct E_8 as above inside a *primitive*, totally real, octic field? Same question for $\mathbb{E}_7$ inside a totally real field of degree 7.

9. Analytic and Algebraic Geometry

This section deals with lattices associated with Abelian varieties over the complex numbers. The problem is to construct such varieties. Jacobian varieties of curves are important examples, with dimension the genus of the curve. However, putting together Hurwitz's and Torelli's theorems, we see that from real dimension 8 onwards, some lattices having a large automorphism groups will not show up.

The question is:
Can one construct "interesting" lattices using the Albanese variety of a variety having a much smaller dimension?

This question is discussed in a 1999 *letter to Eva Bayer and Joseph Oesterlé* (in French), which can be downloaded as a companion file to this file. Note that lattices occur in many constructions in algebraic geometry (Elkies-Shioda theory, ...). Albanese varieties are only one aspect of the question.

10. "Classical" geometry of numbers

For this section, and also for the next one, I refer to [M], Sections 2.5 to 2.7, where I introduced a non-traditional notation, and also to the developments displayed in my homepage: Other texts, *On the Minkowski Constants for Class Groups* (referred to below as [MCCG]) and the other text that follows it (that I intend to turn into a single text), and their common appendix.

Let $A \subset E$ be an open set containing the origin. We say that Λ is *admissible for* A if $\Lambda \cap A = \{0\}$, and define the *lattice constant* $\kappa(A)$ *of* A as the lower bound of the determinants of admissible lattices for A ($\kappa(A) = +\infty$ if no admissible lattices exist). The problem is to calculate (or at least to find good lower bounds for) $\kappa(A)$, and if possible, to describe admissible lattices for A, for suitably chosen sets A which have applications in various domains of number theory. The description of admissible lattices is most of the time out of reach. A particular but important case is the description of *isolated* (admissible) lattices for A, at least for not too large determinants, that is admissible lattices Λ such that a small enough neighbourhood of A only contains admissible lattices of the form $\lambda u(\Lambda)$ with $\lambda \geq 1$ and $u \in \mathrm{Aut}(A)$.

I quote below some important families of domains, among which the Minkowski domains, of a great importance in Algebraic Number Theory.

(1) Quadratic forms. We consider the domains $|q(x)| < 1$ where q is quadratic form of signature (r_1, r_2) with $r_1 + r_2 = n$ (thus, non-degenerate). We may assume that $r_1 \geq r_2$ by negating q. When $r_2 = 0$, the calculation of $\kappa(A)$ amounts to that of γ_n. If $r_2 > 0$ and $n \geq 5$, we have $\kappa(A) = +\infty$ (conjecture of Oppenheim, now a theorem of Margulis). Much (but not all) is known for signature $(1, 1)$. There remains the case when $(r_1, r_2) = (2, 1)$ (resp. $(3, 1)$ or $(2, 2)$), where only a few successive minima are known, by old work of Boris **A**. Venkov[10] (resp. Oppenheim). Domains of signature $(2, 2)$ are related to the theory of indefinite quaternion fields of center $\mathbb{Q}$.
[Besides the Minkowski domains to be considered below, one could also consider domains related to quaternion fields or even Cayley octonions. In the case of totally definite quaternions and octonions with center $\mathbb{Q}$, one find spheres of dimension 4 and 8, with critical lattices produced by the "usual" quaternions ($\mathbb{D}_4$) and octonions ($\mathbb{E}_8$), respectively.]

The subject of the following problem is an analogue to that of Swinnerton-Dyer's paper [SwD].

Problem 10.1. *To make algorithmic and to write down a program extending B.A. Venkov's list of successive minima, and use these data to (try to) guess a putative law analogue to Markoff's law for signature* $(1, 1)$*, but…*
does (by contrast to the Markoff chain) the sequence of successive minima tend to infinity? [My guess is that it does.]

(2) Minkowski domains. With the notation of [M], writing now n as a sum $r_1 + 2r_2$, these are the domains

$$A_{r_1,r_2} = \left\{ x \in E \mid \prod_{i=1}^{r_1} |x_i| \cdot \prod_{j=1}^{r_2} (y_j^2 + z_j^2) < 2^{r_2} \right\},$$

where $y_j = x_{r_1+j}$ and $z_j = x_{r_1+r_2+j}$. We denote by κ_{r_1,r_2} the lattice constant of A_{r_1,r_2}. Thanks to the normalization factor 2^{r_2}, we have in any given dimension the inclusions $A_{r_1',r_2'} \subset A_{r_1,r_2}$ if $r_1' < r_1$, and Minkowski's theorem for class numbers takes the simple form:
Let K be a number field of signature (r_1, r_2) and discriminant d_K. Then any class of ideal of K contains an integral *ideal $\mathfrak{a}$ of norm* $N_{K/\mathbb{Q}}(\mathfrak{a}) \leq \sqrt{|d_K|/\kappa_{r_1,r_2}}$.

The constants κ_{r_1,r_2} are known only if $n = 2$ (and then they coincide with the constants in (1)), and $n = 3$ (Davenport, who found a simple proof for signature $(3, 0)$ but not for $(1, 1)$); see [MCCG]. Minkowski's

[10] Boris **B**. Venkov's father

original bound, obtained by applying his theorem on convex bodies to the largest convex set contained in A_{r_1,r_2}, reads $\sqrt{\kappa_{r_1,r_2}} \ge (\frac{\pi}{4})^{r_2}\,\frac{n^n}{n!}$. Using spheres (as suggested by Minkowski in a letter to Hermite), one obtains $\kappa_{r_1,r_2} \ge (\frac{n}{\gamma_n})^n$, better in low dimensions than the previous one for signatures with a large r_2.

In the applications to Algebraic Number Theory, analytic methods initiated by Zimmert in [Zi] (and a clever use of Weil's "explicit formulae" when only discriminants are concerned) can be used instead of geometric methods to handle class groups. Nevertheless good lower bounds of κ_{r_1,r_2} may prove useful in low dimensions. For $n = 4$, the lower bounds going back to Minkowski yield for $r_2 = 2, 1, 0$, $\kappa \ge 64$, $70.17\ldots$, $113.77\ldots$, to be compared with the expected exact bounds 117, 275, and 725, respectively. The last one has been improved to $\kappa > 500$ by Noordzij (1967; ref [Noo] in [MCCG]). Similarly the constant for $n = 5, r_2 = 0$ has been improved to $\kappa \ge 3251$ by Godwin (1950; ref [God] in [MCCG]).

It would be interesting to extend Swinnerton-Dyer's list of successive minima in [SwD] for the domain of totally real cubic fields, and to try to see whether the sequence of minima looks as if it is unbounded (which is my feeling). Explorations in larger dimensions look intractable. However restricting oneself in dimension 4 to lattices corresponding to quartic fields containing a given quadratic subfield of small discriminant might well be attacked by the methods that Swinnerton-Dyer used in the case of cubic fields; and such a work could be interesting with respect to the problem considered in the next section. [Explicitly we should prove that the lattice constants for lattices containing $\mathbb{Q}(\sqrt{5})$ are 125, 275 and 725 if $r_2 = 0, 1, 2$, respectively, and that the lattice constant for fields containing $\mathbb{Q}(\sqrt{-3})$ is 117 (necessarily, for $r_2 = 0$). Note that (as Sir Peter told me; see the appendix) one should expect Markoff-like phenomena to occur, at least for real quadratic subfields and signature $(4, 0)$.]

11. Diophantine Approximations

One of the most basic problem in the theory of diophantine approximations is, given an irrational number α, to study the smallest constant C for which the inequality

$$|\alpha - \tfrac{p}{q}| < \tfrac{C}{q^2}$$

has infinitely many solutions in coprime integers p, q (or to prove that any positive C is admissible). In [Cas1] this question is transformed into a problem about the minima of real, indefinite quadratic forms, and this last problem has an easy translation in terms of admissible lattices for the Minkowski domain $A_{2,0}$.

The natural generalization to the approximation of several real numbers by rational numbers having a common denominator is also considered in [Cas1]. However the case of dimension $n \geq 2$ sounds different from the one-dimensional case.

In a different direction an extension of the one-dimensional problem consists in considering the approximations of a given complex number by elements of a *given* imaginary quadratic field. This problem was considered in the fifties in (more or less joint) work of Roger Descombes and Georges Poitou; see more specially

[Poi] G. Poitou, *Sur l'approximation des nombres complexes par les nombres des corps imaginaires quadratiques dénués d'idéaux non principaux, particulièrement lorsque vaut l'algorithme d'Euclide*, Ann. Sci. E.N.S. (3) **70** (1953), 199265; MR0066431.

[In this work the notion of a denominator makes sense because the author only considers fields of class number 1. The general case needs an "ideal" modification of the notion of a denominator.]

To evaluate some constants Poitou makes use of the Minkowski constant $\kappa_{0,2}$, more precisely of the lower bound suggested by Minkowski in his letter to Hermite; and indeed only lattices containing a 2-dimensional lattice associated with the embedding of the given imaginary quadratic field need be considered. This is analogue to the usual situation, for which such a restriction does not show up, since it is automatic (the sublattice is $\mathbb{Z} \subset \mathbb{Q}$).

In my opinion the special cases of approximations over $\mathbb{Q}$ or an imaginary quadratic field can be generalized in the following setting.

We consider the following data:

• An integer $n \geq 1$ written as a sum $r_1 + 2r_2$;
• n complex numbers α_i, $1 \leq i \leq n$ which are real for $i \leq r_1$ and satisfy the condition $\alpha_i = \overline{\alpha}_{i-r_2}$ for $r_1 + r_2 + 1 \leq i \leq n$;
• A number field K of signature (r_1, r_2) equipped with embeddings $\sigma_i : K \to \mathbb{C}$ for $1 \leq i \leq n$ with $\sigma_i(K) \subset \mathbb{R}$ for $i \leq r_1$ and $\sigma_i = \overline{\sigma}_{i-r_2}$ for $r_1 + r_2 + 1 \leq i \leq n$,

and we try to find elements $x \in K$ for which the $|\alpha_i - \sigma_i(x)|$ are small. Looking for a generalization of the cases when $(r_1, r_2) = (1, 0)$ or $(0, 1)$, it seems reasonable to associate this approximation problem with the Minkowski domain $A_{2r_1,2r_2}$ and to try to bound the product $\prod_{i=1}^{n} |\alpha_i - \sigma_i(x)|$ in terms of a denominator $\mathrm{dn}(x)$ of x. This could be defined by writing $(x) = \mathfrak{a}\mathfrak{b}^{-1}$ where $\mathfrak{a}, \mathfrak{b}$ are coprime integral ideals and setting $\mathrm{dn}(x) = \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b})$. A precise statement could be that provided the α_i do not lie in the Galois closure of K, the inequality

$$\prod_{i=1}^{n} |\alpha_i - \sigma_i(x)| < \frac{1}{\sqrt{\kappa_{2r_1,2r_2}}} \cdot \frac{1}{\mathrm{dn}(x)^2}$$

should have infinitely many solutions in $x \in K$.

Note that in this statement, K occurs only by its signature. Probably one could replace $\kappa_{2r_1,2r_2}$ by the lattice constant $\kappa_{2r_1,2r_2}(K)$, obtained by restricting oneself to those admissible lattices of dimension $2n$ containing the n-dimension lattice associated with K.

I am no specialist of diophantine problems, and I leave the specialists the task of checking whether the exponent I gave above is reasonable.