FAMILIES OF EQUIANGULAR LINES AND LATTICES

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Abstract. We construct Euclidean lattices whose sets of minimal vectors support some large equiangular families of lines, using notably reduction modulo 2 of lattices. We also consider some related problems, and answer a question raised by Greaves ([G], Subsection 1.3.2).

Introduction

In this note we consider constructions of sets of equiangular lines in a Euclidean space E derived from Euclidean lattices. We postpone to Section 1 the necessary prerequisites on Euclidean lattices together with the statement of various general results, stating here only the following theorem:

Theorem 0.1. *Let Λ be an even (integral) lattice in a Euclidean space E, of minimum m, and let $x_0 \in \Lambda$ of norm (the square of the length) $2m-2$. Then the set vectors of norm $2m+2$ in Λ orthogonal to x_0 and congruent to x_0 modulo 2 supports an equiangular family of lines of common angle* $\arccos \frac{1}{m+1}$.

In the rest of the introduction we solely recall some basic facts and questions concerning families of equiangular lines. Since the publication in 1973 of Lehmens-Seidel seminal paper [Le-S], there has been a large literature on the subject, among which I would like to quote the collective work [JTYZZ] (Ann. Math., 2021). We refer to Greaves' survey [G] for the results mentioned below without a reference. We shall in particular note that for a set of a given rank n, the numbers of lines $t = n$, $n+1$, $2n$ and $\frac{n(n+1)}{2}$ play somewhat special rôles.

Consider a set X of t lines of rank n with common angle $\theta \in [0, \frac{\pi}{2}]$, with which we associate the parameter $\alpha = \arccos\theta$, and assume that $t > n$. We then have $\theta \neq \frac{\pi}{2}$, thus $0 < \alpha < 1$. Equip each line with a

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norm-1 vector, and consider their Gram matrix $G(X)$, with entries 1 on its diagonal and $\pm\alpha$ off its diagonal. The matrix

$$S(X) := \frac{1}{\alpha}(I_n - G(X))$$

is a *Seidel matrix for* X. It depends on the ordering and orientations of the lines in X, but its spectrum solely depends on X, and in particular its smallest eigenvalue λ, equal to $-\frac{1}{\alpha} < -1$. It is immediate that if all $\pm\alpha$ are $+\alpha$ (resp. $-\alpha$), we have $\lambda = -1$, which must be excluded (resp, $\lambda = -(t-1)$, $t = n+1$ and $\alpha = \frac{1}{n}$): in other words, for an acute (resp. obtuse) equiangular family of *vectors*, we have $t \le n$ (resp. $t \le n+1$, with $\alpha = \frac{1}{n}$ if equality holds). Also we see that $t \ge n+1$ implies that the angle may take only finitely many values.

The *absolute bound* (*Gerzon's theorem*) is the inequality $t \le \frac{n(n+1)}{2}$, which holds because the set of orthogonal projections to the line of X is an acute family in $\mathrm{End}^s(E)$ (we have $\mathrm{Tr}(p_x \circ p_y) = (x \cdot y)^2$).

The importance of the value $2n$ (or $2n+1$) comes from *Neumann's theorem*: if $t > 2n$, $\frac{1}{\alpha}$ is an odd integer; compare Theorem 0.1.

In general one considers the maximum number of lines in an equiangular set without taking into account their rank: in terms of the definition below, one considers values or estimations for t_n, *not* for t'_n.

Definition 0.2. Let t_n (resp. t'_n) be the maximal cardinality of an equiangular set contained in an n-dimensional space (resp. spanning an n-dimensional space). [Thus we have $t'_n \le t_n = \max_{k \le n} t'_k$.]

The exact values of t_n are known up to $n = 17$. As we shall see, we have $t'_n = t_n$ for $n \le 7$ and $14 \le n \le 17$, but in the range $7 < n < 14$, we have the *strict inequality* $t'_n < t_n$. It is also known (cf. Neumann's theorem above) that t_n is strictly larger than $2n$ except if $n = 2$, 3 4, 5 or 14. This suggests the following double question:

Question 0.3. *For which dimensions n is it true that $t'_n = t_n$? that $t'_n > 2n$?*

A neighbour question closer to the subject of this note is:

Question 0.4. *For which dimensions is the bound t'_n attained on the set of minimal vectors of a Euclidean lattice ?*

I have put emphasis on t'_n rather than on t_n because I believe that t'_n presents irregularities as a function of n which throw some light on individual properties of dimensions. In this respect it is worth considering the absolute bound $\frac{n(n+1)}{2}$. This is known to be attained for $n = 2, 3, 7, 23$, but on no other dimensions. These dimensions are somewhat special, related to the existence of the regular hexagon and

icosahedron for $n = 2, 3$, and to the lattices $\mathbb{E}_8$ and Leech's Λ_{24} of dimension $n + 1$ for $n = 7, 23$.
[This can be related to the double transitivity of the groups S_3, A_5, $\mathrm{O}_7(2) \simeq S_6(2)$ and Co_3 in degrees $3, 6, 28$ and 276, respectively. For $n = 3$, one identifies the action of A_5 on the diagonals of a regular icosahedron as its action on its 5-Sylow subgroups; lattices for $n = 2, 7, 23$ will be described later.]

Recent works have shown the special behaviour of the "magic" dimensions 8 and 24 for kissing number and sphere packing problems. This suggests that the absolute bound could be strict in all other dimensions. The following question of maximality could be considered in place of it. Consider a configuration X of equiangular lines which is maximal in the n-dimensional space its spans. Say that X is *universally maximal* if X remains maximal when embedded in dimension $n+1$ (or any larger dimension). The four dimensions above are universally maximal: trivially for $n = 2$, because $t'_n = t_n = t_{n+1}$ for $n = 3, 7, 23$.

Question 0.5. *For which dimensions n does there exist a universally maximal configuration of t'_n lines?*

Here are two more results we shall need later. First the *relative bound* ([Li-S], Lemma 6.1; see [G], Th. 1.9): if $t < \frac{1}{a^2}$, then $t \leq \frac{n(1-\alpha^2)}{1-n\alpha^2}$.

Next ([JTYZZ]): if $\alpha = \frac{1}{2k-1}$ $(k \geq 2)$, then we have $t'_n = \lfloor \frac{k(n-1)}{k-1} \rfloor$ for n large enough.

In Section 1 we explain how to apply the theory of lattices modulo 2 as developed in [Ma1] and [Ma2] to the construction of equiangular systems of lines. In Section 2 we consider root lattices (certain lattices of minimum 2) and their duals, and lattices of minimum 3 derived from them, and in Section 3, lattices of minimum 4 and 5.

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1. Basic Results

Let E be a Euclidean space of dimension n, and Λ be a (full) lattice in E. We denote by m its minimum, by S the set of its minimal vectors and set $s = \frac{1}{2}|S|$. More generally, given $r > 0$, let

$$S_r = \{x \in \Lambda \mid x \cdot x = r\} \text{ and let } s_r = \tfrac{1}{2}|S_r|\,;$$

thus, $S = S_m$ and $s = s_m$. We also set $N(x) = x \cdot x$ (the *norm of* x, the square of the usual $\|x\|$). We say that Λ is *integral* if all scalar products on Λ are integral, and then *even* if all norms are even, *odd* otherwise. Note that Λ is integral if and only if it is contained in its dual $\Lambda^* = \{x \in E \mid \forall\, y \in \Lambda,\, x \cdot y \in \mathbb{Z}\}$.

Consider vectors $x, y \in \Lambda$, with $x \neq 0$, $y \neq 0$ and $y \neq \pm x$, and such that $y \equiv x \bmod 2$. Set

$$e = \tfrac{y-x}{2} \text{ and } f = \tfrac{y+x}{2}\,.$$

Then e, f are nonzero, the set $\{\pm e, \pm f\}$ only depends on $\{\pm x, \pm y\}$, and we have

$$x = -e + f \text{ and } y = e + f\,.$$

Moreover, in an equality $\pm y = \pm x + 2u$, u is one of $\pm e$ or $\pm f$.

Proposition 1.1. ([Ma1], § 1) *With the notation above, we have*

(1) $N(x) + N(y) \geq 4m$.
(2) *If* Λ *is integral,* $N(y) \equiv N(x) \bmod 4$.
(3) *If* $N(x) + N(y) = 4m$, *then* e *and* f *are minimal and* $x \cdot y = 0$.

Proof. (1) We have

$$N(x) + N(y) = \tfrac{1}{2}\big(N(y-x) + N(y+x)\big) = 2\big(N(e) + N(f)\big) \geq 4m$$

since e and f are nonzero.

(2) We then have

$$N(y) - N(x) = N(e+f) - N(-e+f) = 4e \cdot f \equiv 0 \bmod 4\,.$$

(3) We then have $N(e) + N(f) = 2m$, hence $N(e) = N(f) = m$ and $x \cdot y = N(f) - N(e) = 0$. □

Proposition 1.1 shows that a list of representatives of classes modulo 2 of smallest possible norms must include vectors of norm $N \leq 2m$, and that the corresponding classes then include a unique pair $\pm x$ if $N < 2m$, and at most n such pairs if $N = 2m$. Among examples for which equality holds are the celebrated lattices $\mathbb{E}_8$ and Λ_{24}.

Proposition 1.2. *Let* $m' \geq m$ *and* $m'' \geq m'$, *and let* $x, y, y' \in \Lambda$ *such that* $N(x) = m'$, $N(y) = N(y') = m''$, $y \equiv y' \equiv x \bmod 2$, *and* $y' \neq \pm y$, *Then* $|y \cdot y'| \leq m'' - 2m$.

Proof. Write as in Proposition 1.1

$$x = -e + f, y = e + f \text{ and } x = -e' + f', y' = e' + f'\,.$$

Calculating $N(y - y')$ in two ways we obtain the equalities

$$2m'' - 2y \cdot y' = 4N(e - e') \geq 4m$$

(since $e' = e$ implies $y' = y$), hence

$$y \cdot y' = m'' - 2N(e - e') \leq m'' - 2m\,,$$

since $N(e \pm e') \geq m$.

Calculating similarly $N(y + y')$, we obtain

$$2m'' - 2y \cdot y' = 4N(-2x + f + f') \geq 4m$$

(since $x = f + f'$ implies $y' = -y$), hence

$$y \cdot y' = m'' - 2N(e - e') \leq m'' - 2m\,.$$

□

Theorem 1.3. *Assume that Λ is integral of even minimum. Let $x_0 \in \Lambda$ of norm $2m-2$. Consider the set $\mathcal{E}(x_0, 2m+2) = \{\pm y_1, \dots, \pm y_k\}$ of those vectors of norm $2m+2$ congruent to x_0 modulo 2. Then the y_i support an equiangular family of lines of angle* $\arccos \frac{1}{m+1}$ *and rank* $\le n-1$. (By an abuse of language, we accept sets of less than three lines.)

Proof. We have $|y_i \cdot y_j| \le 2$ by Proposition 1.2. Next by the calculation done in the proof of this proposition we have $y_i \cdot y_j = m'' - N(e-e')$ with $m'' = 2m+2$, and since $N(e) = N(e') = m$, we have $N(e-e') = 2m - 2e \cdot e'$, which implies

$$y_i \cdot y_j = -2\,m + 2 + 4\,e \cdot e' \equiv 2 \mod 4\,,$$

hence $y_i \cdot y_j = \pm 2$.

The bound for the rank comes from Proposition 1.1, (4). □

Remark 1.4. (1) The restriction to *even* minima is essential. Indeed, if m is odd, we may apply Proposition 1.1 to the even sublattice Λ_{ev} of Λ, of minimum $m_{ev} \ge m+1$. Since $m' + m''$ is strictly smaller than $4(m+1)$, $\mathcal{E}(x_0, 2m+2)$ is then empty.
(2) We may also clearly restrict ourselves to irreducible lattices.

The proposition below, extracted from Section 5 of [Ma2], shows that the set of equiangular lines constructed in Theorem 1.3 can be realized as the set of minimal vectors of an integral (relative) lattice of minimum $m+1$.

Proposition 1.5. *Let $x_0 \in \Lambda$ of norm $m' < 2m$, let $\mathcal{C}$ be its class modulo 2, let $L_0 = \mathcal{C} \cup 2\Lambda$ and let $L = L_0 \cap x_0^{\perp}$. Set $m'' = 4m - m'$, and assume that $\mathcal{E}(x_0, m'')$ is not empty. Then L is a lattice with invariants*

$$\dim L = n-1,\ \min L = m'', \quad \textit{and} \quad S(L) = \mathcal{E}(x_0, m'')\,.$$

Proof. We have $\mathcal{C} = x_0 + 2\Lambda$ and $2x_0 \in 2\Lambda$, hence L_0 is a lattice (containing 2Λ to index 2). By Proposition 1.1, the first minimum of the norm on $\mathcal{C}$ is m', attained uniquely at $\pm x_0$, and since $\mathcal{E}$ is not empty, its second minimum is m", attained exactly on $\mathcal{E}$. Since $\min 2\Lambda = 4m > m''$, these are the first two minima of L_0, and since $\mathcal{E}$ is orthogonal to x_0, we have $\min L = m''$ and $S(L) = \mathcal{E}$.
[The calculation of $\det(L)$ is carried out in [Ma2], Section 5.] □

Remark 1.6. The vectors in L are sums of minimal vectors. This shows that its even part L_{ev} is generated by sums $e + e'$, $e, e' \in S(L)$. Easy calculations then show that on L_{ev}, all scalar products are even and all norms are divisible by 4. Hence $\frac{1}{\sqrt{2}} L_{ev}$ is an even (integral) lattice.

From an algorithmic viewpoint, listing the y_i in Theorem 1.3 from vectors of norm $2m+2$ can need lengthy computations. The following proposition allows us to complete this list using only minimal vectors. This also makes easy the calculation of the rank of $\mathcal{E}$. In the following proposition we keep the notation of 1.3, setting moreover

$$\{S_0 = \{x \in S \mid x_0 \cdot x = m - 1\,.\}$$

Proposition 1.7. *The map*

$$x \mapsto x_0 - 2x \;:\; S_0 \to \Lambda$$

induces a bijection of S_0 *onto* $\mathcal{E}$, *and we have* $\operatorname{rk}\mathcal{E} = \operatorname{rk} S_0 - 1$.

Proof. The first assertion follows from the calculation of $x_0 \cdot \frac{y-x}{2}$ and $N(\frac{y-x}{2})$ for $y \in S_{2m+2}$.

Let $r = \operatorname{rk}\mathcal{E}$, let $y_1, \dots, y_r$ be r independent vectors in $\mathcal{E}$, and let $x_i = \frac{x_0 - y_i}{2}$. We have

$$\langle y_1, ..., y_r\rangle = \langle y_1, y_2 - y_1, \dots, y_r - y_1\rangle = \langle y_1, x_2 - x_1, \dots, x_r - x_1\rangle\,.$$

Now $\langle S_0\rangle$ contains the $x_0 - x_i$ (because $-y_i = x_0 - 2(x_0 - x_i)$) and $x_0 - 2x_1$ $(= y_1)$. Hence $\langle S_0\rangle = \langle x_0, y_1, \dots, y_r\rangle$, and since the y_i but not x_0 are orthogonal to x_0, we have $\operatorname{rk} S_0 = r + 1$. □

Let v be a nonzero vector in E and let $H := (\mathbb{R}v)^\perp$. The orthogonal projection to H (or along v) of $x \in E$ is

$$p(x) = x - \tfrac{v\cdot x}{v\cdot v}\, v\,.$$

Scalar products and norms of projections are given by the formulae

$$p(x)\cdot p(y) = x\cdot y - \frac{(v\cdot x)(v\cdot y)}{N(v)} \text{ and } N(p(x)) = N(x) - \frac{(v\cdot x)^2}{N(v)}\,. \qquad (*)$$

Note that for $x \in S$ and $x \neq \pm v$, we have $|v\cdot x| \leq \frac{N(v)}{2}$, since we have $N(v \pm x) \geq m$, hence $\mp 2(v\cdot x) \leq N(v)$.

Consider a lattice Λ. Then $p(\Lambda)$ is a lattice if and only if $v \in \Lambda$, and we may assume that v is primitive (because p only depends on the line $\mathbb{R}v$). We can then construct bases $(v_1, \dots, v_n)$ for Λ with $v_1 = v$, so that $(p(v_2), \dots, p(v_n))$ is a basis for $p(\Lambda)$.

In the setting of Theorem 1.3 projections along x_0 preserve the norms on $\mathcal{E}$, and shall be used to construct lattices whose minimal vectors support equiangular families of lines. However I cannot give *a priori* a general procedure to find minimal vectors in projections. Note that if $|v\cdot x|$,takes values $\alpha_1 < \dots < \alpha_t$ on $S(\Lambda) \smallsetminus \{\pm v\}$, the corresponding norms of the projections occur in the reverse order, and $\alpha_t = \frac{N(v)}{2}$ needs $N(v) < 4m$ by $(*)$. We shall consider the projections of the $x \in S(\Lambda)$ such that $v\cdot x = \pm\alpha_t$, in particular when $N(v) = 2m-2$ and $\alpha_t = \frac{N(v)}{2}$, since we then obtain equiangular families of line by applying Theorem 1.3. It will often happen that $S(p(\Lambda))$ is the subset of $p(S(\Lambda))$ of the $p(x)$ with $v \cdot x = \frac{N(v)}{2}$. This will be checked for all lattices we shall construct in the forthcoming sections.

We state below a proposition which gives us some precisions on projections. Given v we consider a partition $S^+ \cup S^-$ of S for which $x \in S^+$ needs $v \cdot x \geq 0$ (the choice of $x \in S^+$ among $\pm x$ is well-defined only when $v \cdot x \neq 0$). Using formulae $(*)$, we easily prove:

Proposition 1.8. *Consider a lattice Λ of minimum m, a vector $v \in \Lambda$ of norm $2m-2$ and the orthogonal projection p along v, and assume that $S(p(\Lambda))$ is contained in $p(S(\Lambda))$. Let x and $y \neq \pm x$ in S^+, not colinear with v. Then we have*

$$\frac{m-3}{4} \leq x \cdot y \leq \frac{3m-1}{4}\,.$$

In particular if $m = 2$ (resp. $m = 4$) we have $x \cdot y \in \{0,1\}$ (resp. $x \cdot y \in \{1,2\}$. □

2. Lattices of minimum 2 or 3

Root lattices are integral lattices generated by norm-2 vectors. These are orthogonal sums of irreducible lattices, isometric to exactly one of $\mathbb{A}_n$ $(n \geq 1)$, $\mathbb{D}_n$ $(n \geq 4)$ or $\mathbb{E}_n$ $(n = 6,7,8$, the definition of which we recall below; see [Ma], Chapter 4 for details.

Inside the lattice $\mathbb{R}^{n+1}$ (resp. $\mathbb{R}^n$), equipped with its canonical basis $(\varepsilon_0,\dots,\varepsilon_n)$ (resp. $(\varepsilon_1,\dots,\varepsilon_n)$, Let

$$\mathbb{A}_n = \{x \in \mathbb{Z}^{n+1} \mid \textstyle\sum_{i=0}^{i=n} x_i = 0\} \text{ and } \mathbb{D}_n = \{x \in \mathbb{Z}^n \mid \textstyle\sum_{i=1}^{i=n} x_i \equiv 0 \mod 0\}\,.$$

In $\mathbb{R}^8$ let $e = \frac{\varepsilon_1+\cdots+\varepsilon_8}{2}$, set $\mathbb{E}_8 = \mathbb{D}_8 \cup (\mathbb{D}_8 + e)$, and define $\mathbb{E}_7$ and $\mathbb{E}_8$ by successive sections orthogonal to $\varepsilon_7 - \varepsilon_8$ and $\varepsilon_6 - \varepsilon_7$.

For a lattice Λ in the list above the automorphism group acts transitively on its set of minimal vectors, among which we may choose arbitrarily x_0 when applying Theorem 1.3. Let us choose $x_0 = \varepsilon_0 - \varepsilon_1$, $\varepsilon_1 - \varepsilon_2$ and e when $\Lambda = \mathbb{A}_n$, $\mathbb{D}_n$ and $\mathbb{E}_n$, respectively.

For $\Lambda = \mathbb{A}_n$, we have

$$S_0 = \{-\varepsilon_0 + \varepsilon_i, \varepsilon_1 - \varepsilon_i,\, i \geq 2\} \quad\text{and}\quad \pm\,\mathcal{E} = \{\varepsilon_0 + \varepsilon_1 - 2\varepsilon_i,\, i \geq 2\}\,.$$

For $\Lambda = \mathbb{D}_n$, we have

$$S_0 = \{-\varepsilon_1 \pm \varepsilon_i, \varepsilon_2 \pm \varepsilon_i,\, i \geq 3\} \quad\text{and}\quad \pm\,\mathcal{E} = \{\varepsilon_1 + \varepsilon_2 \pm \varepsilon_i,\, i \geq 3\}.$$

For $\Lambda = \mathbb{E}_8$, S_0 consists of $28 \times 2 = 56$ vectors, the $\varepsilon_i + \varepsilon_j$, $1 \leq i < j \leq 8$ and the 28 vectors obtained by negating 6 basis vectors in e. The 28 vectors in $\mathcal{E}$ (up to sign) are then obtain by permutations of

$$\frac{3\varepsilon_1+3\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_6-\varepsilon_7-\varepsilon_8}{2}\,.$$

The results for $\mathbb{E}_7$ and $\mathbb{E}_6$ are obtained by sections of $\mathbb{E}_8$. Alternatively we could also have used the sets of minimal vectors in projections of $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_n$.

Summarizing, we obtain:

Proposition 2.1. *Let* Λ *be an irreducible root lattice of dimension* n*. Then the equiangular family of Theorem 1.3 is of rank* $r = n-1$ *and contains* t *lines according to the following data:*

$$\Lambda = \mathbb{A}_n : t = r\,; \quad \Lambda = \mathbb{D}_n : t = 2r-2\,;$$
$$\Lambda = \mathbb{E}_8 : t = 28\,; \quad \Lambda = \mathbb{E}_7 : t = 16\,; \quad \Lambda = \mathbb{E}_6 : t = 10.$$

[$\mathbb{D}_{n+1}$ provides the asymptotic bound for $\alpha = \frac{1}{3}$, valid for all $n \geq 15$.] □

As for the duals, we may discard $\mathbb{D}_4^* \sim \mathbb{D}_4$, $E_8^* = E_8$ and $\mathbb{D}_n^*$, $n \geq 5$ (because $S(D_n^*) = S(\mathbb{Z}^n)$). The set $S(\mathbb{A}_n^*)$ supports the equiangular set of *vectors* with common angle $-\arccos\frac{1}{n}$, and $S(\mathbb{E}_7^*)$ is is the projection of $\mathcal{E}$ attached to $\mathbb{E}_8$. The lattice $\mathbb{E}_6^*$ does not apply directly to equiangular systems of lines.

[The function $|x \cdot y|$ takes two values on non-proportional $x, y \in S(\mathbb{E}_6^*)$. We shall return to such lattices in [Ma3] in connection with the theory of *strongly regular graphs* (with $S(\mathbb{E}_6^*)$ we can recover the *Schläfli graph*, attached to the system of lines on a non-singular cubic surface).]

The values of t_n are known up to $n = 17$ (see [G]). For $n = 2$ to 7, t_n is equal to $3, 6, 6, 10$ and 16, respectively, and we have $t_n = 28$ for $7 \leq n \leq 14$.

Proposition 2.2. (1) *We have* $t'_n = t_n$ *for* $2 \leq n \leq 7$*, and except for* $n = 3$*,* t'_n *is attained on the set of minimal vectors of a lattice.*

(2) *For* $n = 3$*, the maximum number of lines defined by a lattice is* 4*, attained uniquely on the configuration of* $S(\mathbb{A}_3^*)$*.*

Proof. (1) This is clear for $n = 2$ and 3. For $n = 4, 5, 6, 7$, consider the projections of $\mathbb{D}_5$, $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ (Proposition 2.2).

(2) This results from the classification of *minimal classes* for $n = 3$ (see [Ma], Theorem 9.2): one checks that a lattice with $s \geq 4$ must have a system of minimal vectors containing $e_1, e_2, e_3, e_1 + e_2 + e_3$ with equal scalar products $e_i \cdot e_j$. □

Another way to construct lattices of minimum 3 consists in viewing them as lattices containing to index 2 their even sublattice. For further use we consider more generally lattices of minimum $m \geq 3$ odd. To state the results below we introduce some notation. Given a lattice L and $a > 0$, we denote by ${}^aL$ the group L equipped with the scalar product $a\,(x \cdot y)$. (Thus we have ${}^aL \simeq \sqrt{a}\,L$.) Given an integral lattice L, L_{ev} denotes the *even part of* L, i.e., the set of $x \in L$ having an even norm. If L is odd, we have $[L : L_{ev}] = 2$ and $L = \langle L_{ev}, e \rangle$ where $e \in L$ is any vector of odd norm.

Proposition 2.3. *Let Λ be a lattice of dimension $n \geq 2$ and of odd minimum $m \geq 3$, generated by k minimal vectors e_i of pairwise scalar products ± 1. Then $L := {}^{1/2}\Lambda_{ev}$ is an even lattice of minimum $m' \geq \frac{m}{2}$, and either we have $s(\Lambda) \leq n$, or L has a basis of vectors of norm $m-1$.*

Proof. First note that Λ_{ev} is generated by the sums $x+y$, $x, y \in S(\Lambda)$, with $y \neq x$ since $2x = (x+y)+(x-y)$. We have $N(x+y) = 2(m \pm 1) \equiv 0 \bmod 4$, which shows that L is even.

Choose a half-system $e_1, \dots, e_k$ such that $e_1 \cdot e_i = +1$ for $i \geq 2$ and that $e_1, \dots, e_n$ are independent, and set $e'_i = e_1 - e_i, i \geq 2$. Suppose first that all $e_i \cdot e_j, i < j$ are equal to $+1$ and that $s > n$. Then we may write $x := e_{n+1}$ as a $\mathbb{Q}$-linear combination of $e_1, \dots, e_n$, say, $x = \sum \lambda_k \, e_k$. Calculating $x \cdot e_i$ for $i \leq n$, we obtain

$$1 = x \cdot e_i = m\lambda_i + \textstyle\sum_{i \neq j} \lambda_j = (m-1)\lambda_i + \sum_j \lambda_j \,,$$

which implies that the λ_i have the common value $\lambda = \frac{1}{n+m-1}$.

From $x = \lambda \sum e_i$ we deduce that $m = x \cdot x = \lambda \, n$, which implies

$$\tfrac{1}{n+m-1} = \tfrac{m}{n} \;\Leftrightarrow\; (m+n)\,(n-1) = 0 \,,$$

a contradiction.

[Lattices with $e_i \cdot e_j = +1$ and $s = n$ exist, and are unique up to isometry. They can be represented by the Gram matrix M with entities $M_{i,i} = m-1$ and $M_{i,j} = \frac{m-1}{2}$ if $j \neq i$, *except* $M_{1,1} = m+1$ and $M_{1,2} = M_{2,1} = 0$.]

Otherwise, let $i > 1$ and $j > i$ such that $e_i \cdot e_j = -1$. If $i > n$, write e_i as a Q-linear combination of $e_\ell, \ell \leq n$. There must be at least three nonzero components, so that exchanging i with some $i' \leq n$, me may assume that $i \in [2, n]$, that me may then exchange with 2. We thus reduce ourselves to the case when $i = 2$, and then we may similarly assume that $j = 3$. Finally we check that $e_2 + e_3, e'_2, \dots, e'_n$ is a basis for Λ_{ev}, which completes the proof of the dichotomy between the two cases above. □

Note that given L we can reconstruct Λ from L by the formula

$$\Lambda = {}^2L_1 \text{ with } L_1 = \langle L, \tfrac{v}{2} \rangle \text{ and } v \in L \text{ of norm } 2m \,.$$

Theorem 2.4. *Let E be a Euclidean space of dimension $n \geq 8$, and let X be a spanning family of s equiangular lines of angle* $\arccos \frac{1}{3}$*. Then we have $s \leq 2n-2$, and equality holds uniquely on the equiangular set obtained by applying Proposition 2.1 to the root lattice $\mathbb{D}_{n+1}$.*

Proof. We write below a simple direct proof, based on ideas of Section 4 of [Le-S], but without making use of calculations with matrices. We denote by $X = \{e_1, \dots, e_s\}$ a set of s vectors representing the pairs $\pm x$.

Let k be the largest integer such that there exist vectors $e_1, ..., e_k$ of norm 1 and pairwise scalar products $\pm\frac{1}{3}$. We first show that we have $k = 4$.

Indeed, we have $k > 0$, hence $k \ge 2$, since vectors e_i with all $e_i \cdot e_j \ge 0$ are independent, which implies $s \le n \le 2n-2$ if $n \ge 3$. Next the lower bound $\mathrm{N}(e_1 + ... + e_k) \ge 0$ implies $k \le 4$, and $k = 3$ is excluded because $\{e_1, e_2, e_3\}$ can be enlarged with $e_4 := -(e_1 + e_2 + e_3)$.

Suppose that $k = 2$. Then the e_i may be assumed to be such that $e_1 \cdot e_2 = -\frac{1}{3}$ and $e_1 \cdot e_i = +\frac{1}{3}$ for $i \ge 3$. Then we have $e_2 \cdot e_i = -\frac{1}{3}$ for $i \ge 3$ (otherwise we would have $k \ge 4$ by a set $\{e_1, e_2, -e_i\}$). If $e_i \dot{e}_j = -\frac{1}{3}$ for some $i \ge 3$ and $j > i$, then we have $k \ge 4$, as shown by the set $\{e_2, e_i, e_j\}$.

From now on, $k = 4$. Thus X contains e_1, e_2, e_3 and $e_4 := -(e_1 + e_2 + e_3)$, and only the $\pm e_i$. Denote by $\Pi \subset E$ their span (of dimension 3). Choose the e_i, $i \ge 5$ such that $e_4 \cdot e_i = +\frac{1}{3}$. Let $x \in X \smallsetminus \Pi$, and let h (a "pillar") be its projection to Π. We have $h \cdot e_i = x \cdot e_i = \pm\frac{1}{3}$ and $\sum h \cdot e_i = 0$. so that there are at most three pillars $h_i, i = 1, 2, 3$, characterized by $h \cdot e_i = +\frac{1}{3}$ for some $i \in \{1, 2, 3\}$. This shows that we have $h_i = \frac{e_i + e_4}{3}$, and consequently,

$$\mathrm{N}(h_i) = \tfrac{1}{3} \ \text{ and } \ h_i \cdot h_j = 0 \ \text{ for } j \ne i\,.$$

We prove now that if two pillars are involved, then n is ≤ 7. Consider two pillars h, h', let $\mathcal{L}$ (resp. $\mathcal{L}'$) be the span of $c \in \Pi^\perp$ such that $h + c \in X$ (resp. $h' + c \in X$) and set $\ell = \dim \mathcal{L}$. For $c \in \mathcal{L}$ and $c' \in \mathcal{L}'$, we have $c \cdot c' = x \cdot x' = \pm\frac{1}{3}$. Write $c' = \sum u_i c_i$ on an orthogonal basis of $\Pi^\perp$ extending an orthogonal basis $(c_1, \dots, c_l)$ of $\mathcal{L}$. We have $c' \cdot c_i = \pm\frac{1}{3}$ for $i \le \ell$, hence $u_i = \frac{1}{2}$ for $i \le \ell$, and $\mathrm{N}(c') \ge \frac{\ell}{4} \cdot \frac{2}{3}$. Since $N(c') = \frac{2}{3}$, we have $\ell \le 4$. Moreover, $c' \mapsto u_i$ defines a map $\mathcal{L}' \to \mathcal{L}$ which is onto. Exchanging h and h' shows that $\mathcal{L} = \mathcal{L}'$, whence $n = 3 + \ell \le 7$. This completes the proof of the upper bound $s \le 2n - 2$.

The existence of a configuration of lines of angle $\arccos\frac{1}{3}$ is shown by applying Proposition 2.1 to the root lattice $\mathbb{D}_{n+1}$: let $\varepsilon_1, \dots, \varepsilon_{n+1}$ be the canonical basis of $\mathbb{Z}^{n+1}$; then $x_0 := \varepsilon_n - \varepsilon_{n+1}$ is a minimal vector of its even sublattice $\mathbb{D}_{n+1}$ and the congruence $x \equiv x_o \mod 2$ among norm 6 vectors holds precisely on the set $\{\pm(\varepsilon_n + \varepsilon_{n+1} + \pm 2\varepsilon_i)\}$ with $i = 1, \dots, n$.

[As a Gram matrix for the projection of $\mathbb{D}_{n+1}$ we may choose $A = (a_{i,j})$ with entries 3 on the diagonal and 1 off the diagonal, except $a_{1,2} = a_{2,1} = -1$.]

There remains to prove uniqueness. We have seen that for $n \ge 4$ and $s > n + 1$, Π contains a system of 4 vectors ε_i adding to 0. Then $\{\pm e_i\}$ is similar to $S(\mathbb{A}_3^*)$, whose automorphism group acts transitively on the ε_i and on the pillars. When $n \geq 8$, $X \smallsetminus S(\mathbb{A}_3^*)$ is a collection of $h + c$, for h a fixed pillar, and c runs through an orthogonal frame in $\Pi^\perp$. Since these frames are acted on transitively by $\mathrm{O}(\Pi^\perp)$, there exists a unique isometry class of $2n - 2$ vectors up to sign. □

Here are a few remarks.

(1) The proof above shows more: if $n \geq 8$ and $s > n + 1$, a set of equiangular lines with angle arccos $\frac{1}{3}$ is contained in the set defined by $\mathbb{D}_{n+1}$.

(2) If $n \leq 7$ we have $s \leq 4 + (6n - 3)$, hence $s \leq 28$ if $n = 7$. This bound is optimal (and attained on $S(\mathbb{E}_7^*)$), but finding the optimal bounds for lower n needs some more work.

(3) We have seen that for $n \geq 8$ the bounds for lines with angle arccos $\frac{1}{3}$ are attained on minimal vectors of a lattice. The same is true for $n \leq 7$ (on sections of $\mathbb{E}_7^*$), but with non-optimal s if $n = 2$ or 3.

Our knowledge of low dimensions shows that we always have $t'_n = t_n$ for all $n \leq 7$. This is no longer true for $8 \leq n \leq 13$ (where $t_n = t_7 = 28$). The theorem below gives some precision about dimensions 8 to 13.

Theorem 2.5. *We have* $2n - 2 \leq t'_n \leq 2n$ *for* $8 \leq n \leq 12$ *and* $t'_n = 2n = 26$ *for* $n = 13$.

Proof. Theorem 2.4 shows that we have $g'_n \geq 2n - 2$ for any $n \geq 8$, and we shall construct in Section 3 an integral lattice of minimum 5 showing $t'_{13} \geq 26$.

In the other direction, we know by Newman's theorem that we have $s \geq 2n$ except possibly for an angle equal to arccos $\frac{1}{p}$ for some odd $p \geq 3$. Theorem 2.4 excludes $p = 3$, and the relative bound (see Section) shows that for $n < 25$, the largest value is obtained with $p = 1/5$ and is equal to $\frac{24n}{25-n} \leq 2n$ if $n \leq 13$. □

[The bound above with $p = 5$ gives us $t_{15} = t'_{15} = 36$, but is not sharp enough for $n = 14$ (where $t_{14} = t'_{14} = 28$). In both cases examples exist with angle arccos $\frac{1}{5}$.]

3. Lattices of minimum 4 and 5

In this section we concentrate on lattices of minimum 5 in which pairs of non-proportional minimal vectors have scalar product ± 1. In dimensions $n \in [15\text{–}23]$ (and in some higher dimensions) examples show that we have the strict inequality $t'_n > 2n$, so that the highest values of t'_n are attained on sets of angle arccos $\frac{1}{5}$, whence the special interest of these lattices.

Most of the lattices we shall consider (though not all) have been constructed using projections of a lattice of minimum 4 along a norm-6 vector, and using descending chains of cross-sections. We first consider such an exception.

3.1. **A 13-Dimensional Lattice.** This is the lattice L denoted by $C2 \times \mathrm{PSL}(2,25) : C2$ in [WebN-S] (after its automorphism group). It has $s = 26$, and defines the equiangular family referred to in Theorem 2.5. It has a unique (up to isometry) 14-dimensional extension $\overline{L}$ with $s = 28$, the value of t_{14}, which extends uniquely exactly up to dimension 19 under the condition that, at each step, s should be as large as possible. We recover these lattices as the unique sequence of densest sections, for which Gram matrices denoted by Qan can be read in [WebM], file Min5.GP.
[The even sublattice L_{ev} of L deserves a remark: this is the best known example (found by Conway and Sloane, [C-S]) of a lattice in its dimension having a large "Bergé-Martinet invariant" γ', defined by $\gamma'(\Lambda) = \big(\gamma(\Lambda)\gamma(\Lambda^*)\big)^{1/2}$.]

3.2. **Lattices from the Leech Lattice.** There is a unique orbit of norm-6 vectors in the Leech lattice Λ_{24}, so that Theorem 1.3 defines a unique lattice (up to isomorphism). Its set of minimal vectors consists in 276 pairs $\pm x$, which are the projections along the corresponding norm-6 vector of vectors $x \in S(\Lambda_{24})$ such that $v \cdot x = \pm 3$. This configuration is known as *the Witt design*. Its automorphism group is $2 \times \mathrm{Co}_3$.

Consequently, for any pair (L, v) of a relative lattice $L \subset \Lambda_{24}$ of minimum 4 and of a norm-6 vector $v \in L$, provided that there exists an $x \in S(L)$ with $v \cdot x = \pm 3$, projection along v defines a lattice whose set of minimal vectors supports an equiangular family contained in the Witt design.

We have considered projections of various lattices contained in the Leech lattice in dimensions $14-24$ taken from [WebM], file Lambda.gp. The best results were obtained using Conway-Sloane's *laminated lattices* Λ_n or successive sections of such lattices. The file Min5.gp contains series of Gram matrices Qbn, $16 \ge n \ge 8$ and Qcn, $23 \ge n \ge 8$, starting with Gram matrices for projections of Λ_{17} and Λ_{24}, respectively.

The table below displays the known values of t_n (taken from [GSY] and the largest values found on lattices for equiangular families, using Qan and Qbn for $n = 14$, Qbn for $n = 15, 16$, and Qcn for $17 \le n \le 23$.

Table for dimensions 14 – 23

n	14	15	16	17	18	19	20	21	22	23
t_n	28	36	40	48	57 – 59	72 – 74	90 – 94	126	176	276
$lat \geq$	28	36	38	48	56	72	90	126	176	276

Inspection of the second line of this table clearly shows that we have $t'_n = t_n$ for all $n \in [14, 23]$, and inspection of the third line shows that this values are attained on minimal vectors of lattices for $n = 14$, 15, 17 and 21–23, As for the remaining values of n I conjecture:

Conjecture 3.1. *For $n = 16$, 18, 19 and 20, the largest number of lines produced by the minimal vectors of a lattice is 38, 56, 72 and 90, respectively.*

We now prove two complements, first for $n = 14$, then for $n = 18$, the latter answering a question raised in [G], Subsection 1.3.2.

Proposition 3.2. *The sets of equiangular lines afforded by $S(\mathrm{Qa}\,14)$ and $S(\mathrm{Qb}\,14)$ are not isometric.*

Proof. The construction of $S(\mathrm{Qa}\,14)$ from $S(\mathrm{Qa}\,13)$ shows that removing two convenient lines from $S(\mathrm{Qa}\,14)$ produces a set of rank 13 (indeed, in a unique way), whereas one must remove at least four lines from $S(\mathrm{Qb}\,14)$) to obtain a set of rank ≤ 13. □

In [GSY] the authors construct four sets of 57 vectors which, when rescaled to norm 5 have pairwise scalar products ± 1. I have checked that in all examples the sublattice they generate in $\mathbb{Z}^{18}$ has a minimum $m \leq 4$ (and contains norm 5 vectors with pairwise scalar products not ± 1). Since any subset of norm 5 vectors in Qa 23 generate a lattice of minimum 5 in its span, we have:

Proposition 3.3. *The four equiangular systems above are not contained in the Witt design.* □

3.3. **Beyond dimension 23.** The arguments used to prove Proposition 2.2, (3) (relying on results of [K-Y] together with the relative bound) prove:

Proposition 3.4. *In the range $24 \leq n \leq 41$, t'_n is strictly smaller than t_n (equal to 276).* □

In analogy with what was observed in Section 2 for angle $\arccos \frac{1}{3}$, we expect that $\arccos \frac{1}{5}$ should play a major rôle in dimensions, say, 24 to 50. However, whereas lattices contained in the Leech lattice constitute a rich source of even lattices of minimum 4, our knowledge of such lattices in larger dimensions is poor. I have carried out some experimentation (far from being exhaustive) on the even sublattices of unimodular lattices of minimum 3, using the Bacher-Venkov classification in dimensions 27 and 28 (see the file unimod23to28.gp.gp in[WebM]). Using projections of the even part of the lattice denoted there by $o27b1$ we found lattices with $(n, s) = (26, 82)$ and $(25, 108)$, and then $(n, s) = (24, 100)$ by cross-sections. Gram matrices can be downloaded from Part 4 of Min5.gp.

Also no infinite series of lattices having a fixed minimum $m \geq 4$ are known (by contrast with minima $m = 2$ or 3 for which we can use root lattices). This leaves wide open the following question:

Question 3.5. *Do there exist for each odd $m \geq 5$ infinite series L_n of lattices with minimum m and pairwise scalar products ± 1 on $S(L_n)$ such that $s \sim \frac{m+1}{m-1}\, n$ for $n \to \infty$? (If not the exact bound $\lfloor \frac{m+1}{m-1}\,(n-1) \rfloor$ of* [JTYZZ]*.)*

I have also considered angles $\arccos \frac{1}{7}$, using projections of lattices of minimum 6. The only example which deserves to be mentioned is due to G. Nebe ([Ne], answering a question of mine), the proof of which relies on the theory of modular forms: *applying Theorem 1.3 to an even, unimodular lattice of minimum* 6 *yields an equiangular family of* 100 *lines*.

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