**Classification of positive forms having prescribed automorphisms**

by A.-M. Bergé

## 1. Introduction

Fix an integer $n \geq 2$, and a finite subgroup $G$ of $\mathrm{GL}_n(\mathbb{Z})$. We denote by $\mathcal{S}_G$ the space of real $n \times n$ symmetric matrices $A$ which are $G$-*invariant*, i. e. such that

$$^{t}gAg = A \quad \text{for all } g \in G.$$

The centralizer $\mathrm{SL}_G(\mathbb{Z})$ of $G$ in $\mathrm{SL}_n(\mathbb{Z})$

$$\mathrm{SL}_G(\mathbb{Z}) = \{u \in \mathrm{SL}_n(\mathbb{Z}) \mid g^{-1}ug = u \text{ for all } g \in G\}$$

acts on $\mathcal{S}_G$ by equivalence, and we say that $A$ and $B \in \mathcal{S}_G$ are $G$-*integrally equivalent* if there exists $u \in \mathrm{SL}_G(\mathbb{Z})$ such that $^{t}uAu = B$.

For a positive definite matrix $A \in \mathcal{S}_G$ (there exist such matrices, for instance $\sum_{g \in G} {}^{t}gg$) we denote by $\min A = \min({}^{t}\boldsymbol{x}A\boldsymbol{x},\ \boldsymbol{x} \in \mathbb{Z}^n \smallsetminus \{0\})$ the arithmetic minimum of the associated quadratic form, and by $S(A)$ the set of its minimal vectors (for which ${}^{t}\boldsymbol{x}A\boldsymbol{x} = \min A$). We denote by $\mathcal{P}_G \subset \mathcal{S}_G$ the variety of positive definite $A$'s in $\mathcal{S}_G$ scaled so as to have minimum $\min A = 1$. We classify these matrices by their minimal vectors. More explicitly, the $G$-*cell* of $A \in \mathcal{P}_G$ is the set of matrices $M \in \mathcal{P}_G$ such that $S(M) = S(A)$; this is a convex polyhedron, bounded if and only if the set $S(A)$ spans $\mathbb{R}^n$. This cellular decomposition of $\mathcal{P}_G$, first introduced by Štogrin in [S] for the usual case (i.e. when $G$ is trivial), was extended to the general case by J. Martinet and the author (cf. [BM2]). We also proved in [BM2] that up to $G$-integral equivalence, there are only finitely many bounded $G$-cells. In section 2 of the present paper we make explicit the splitting of a class of integrally equivalent $G$-matrices into several classes of $G$-equivalence, and we give the "mass formula with signs" for bounded $G$-cells due to C. Bavard ([Ba]). The remainder of this paper is devoted to the case of the regular representation of a cyclic group. Section 3 gives a description of the corresponding arithmetic groups, and their Euler characteristic. Tables in section 4 show the complete classification up to dimension 5, providing checks on Bavard's formula.

To describe a given $G$-cell, we generally use an outstanding representative connected to the study on this cell of the Hermite function

$$\gamma(A) = \min A\,(\det A)^{-1/n}.$$

Let $A \in \mathcal{P}_G$ be a $G$-invariant matrix: its inverse $A^{-1}$ as well as the matrices $(\boldsymbol{x}\,{}^{t}\boldsymbol{x},\ \boldsymbol{x} \in S(A))$ are invariant under the group ${}^{t}G = \{{}^{t}g,\ g \in G\}$. We say that $A$ is *eutactic* (resp. *weakly eutactic*) if $A^{-1}$ belongs to the convex hull (resp. the linear span) of $(\boldsymbol{x}\,{}^{t}\boldsymbol{x}, \boldsymbol{x} \in S(A))$. In a given $G$-cell $c$, there exists at most one matrix $A$ where the strictly log-convex Hermite function $\gamma$ attains its minimum; in a joint work with Martinet ([BM3]) we characterized this $A$ in its cell by the property of weak eutaxy. Note that if $A$ is weakly eutactic, the set $S(A)$ spans the space $\mathbb{R}^n$. So, in order to enumerate the weakly eutactic $G$-matrices, we only need classify bounded cells. Among the weakly eutactic $G$-matrices, the eutactic ones and the $G$-perfect ones (whose cells have affine dimension 0) are of particular interest, as eutaxy plus $G$-perfection characterizes the local maxima on $\mathcal{P}_G$ of the Hermite function ([BM1]). In the case of the cyclic group investigated in sections 3 and 4, we note that $G$-perfect matrices are indeed perfect, except for one quintic matrix (which first appeared in [BM1]), and that the root lattice $\mathbb{D}_4$ does not appear since it does not afford the right representation.

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## 2. Euler formula for $G$-cells

Let $c$ be a bounded $G$-cell, and $S$ the set of minimal vectors of the matrices $A \in c$. By assumption, $S$ spans $\mathbb{R}^n$, so the stabilizer $\operatorname{Aut}_G c$ of $c$ in $\mathrm{GL}_n(\mathbb{Z})$, which also stabilizes $S$, is finite.

**Lemma.** *Let $k = |\operatorname{Aut}_G c|$ be the order of this group. Then up to $G$-integral equivalence there are at most $k^{|G|}$ $G$-cells integrally equivalent to $c$.*

Proof. Let $u \in \mathrm{SL}_n(\mathbb{Z})$. Then the group $G$ acts on the cell ${}^t u c u$ if and only if $ugu^{-1}$ belongs to $\operatorname{Aut}_G c$ for all $g \in G$. On the other hand, if $u$ and $v$ in $\mathrm{SL}_n(\mathbb{Z})$ satisfy $ugu^{-1} = vgv^{-1}$ for all $g \in G$, then $v^{-1}u$ belongs in the centralizer $\mathrm{SL}_G(\mathbb{Z})$, and the cells ${}^t u c u$ and ${}^t v c v$ are $G$-integrally equivalent.

*Remark.* The estimate $k^{|G|}$ is very rough, and can be replaced by $k$ in the cyclic case, and by 2 in the cases of section 4.

In the following, we denote by $\operatorname{Aut}_G^+ c = \{u \in \mathrm{SL}_G(\mathbb{Z}) \mid {}^t u c u = c\}$ the stabilizer in $\mathrm{SL}_G(\mathbb{Z})$ of the $G$-cell $c$, and by $\dim c$ its affine dimension.

**Theorem (Bavard).** *Let $\mathcal{B}_G$ be a full set of representatives of the bounded $G$-cells modulo $G$-integral equivalence. Then*

$$\sum_{c \in \mathcal{B}_G} \frac{(-1)^{\dim c}}{|\operatorname{Aut}_G^+ c|} = \chi(\mathrm{SL}_G(\mathbb{Z})),$$

*where $\chi$ stands for the Euler characteristic.*

In the case of the whole set of positive matrices, this formula has been checked up to dimension 5, together with the Ash formula (cf. [A]) for eutactic matrices:

for $2 \le n \le 4$ there are 25 inequivalent cells, all of them but one with a weakly-eutactic representative (cf. [BM2]). Beyond, the combinatorics seemed untractable, but C. Batut recently developped in [B] an algorithm, which produced 136 cells in dimension 5, among which 127 contain a weakly eutactic representative.

## 3. Application to the case of the regular representation of the cyclic group of order $n$

The group $G$ is generated by the permutation matrix

$$\pi = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix};$$

the $G$-invariant matrices are *circulant* matrices (i.e. with a constant diagonal and whose rows are deduced from the first one by circular permutation). The space $\mathcal{S}_G$ of the symmetric invariant matrices is of dimension $1 + \lfloor n/2 \rfloor$, as the first row $(a_1, a_2, \cdots, a_n)$ of such a matrix satisfies $a_i = a_j$ if $i + j = n + 2$.

We now discuss the Euler characteristic of the centralizer $\mathrm{SL}_G(\mathbb{Z})$ of $G$ in $\mathrm{SL}_n(\mathbb{Z})$. We denote by $I_n$ the unit matrix in $\mathrm{SL}_n(\mathbb{Z})$. For even $n$, we denote by $G^+$ the subgroup of $\mathrm{SL}_n(\mathbb{Z})$ generated by $\pm I_n$ and $\pi^2$.

**Proposition.**

(1) $\chi(\mathrm{SL}_G(\mathbb{Z})) = \begin{cases} \dfrac{1}{n} & \textit{if} \quad n = 2, 3, 4 \ \textit{or}\ 6, \\ 0 & \textit{if} \quad n = 5 \ \textit{or}\ n \ge 7. \end{cases}$

(2) $\mathrm{SL}_G(\mathbb{Z}) = G^+$ *(resp. $G$) for $n = 2, 4, 6$ (resp. $n = 3$). For $n = 5$, $\mathrm{SL}_G(\mathbb{Z})$ is generated by $G$ and the circulant matrix $\sigma$ with first row $(-1, 0, 1, 1, 0)$.*

(3) *Let $c$ be a bounded $G$-cell. Then $\mathrm{Aut}_G^+ c = G^+$ (resp. $G$) for $n = 2, 4, 6$ (resp. $n = 3, 5$).*

*Proof.* The group $\mathrm{SL}_G(\mathbb{Z})$ is abelian, and has positive Euler characteristic if and only if it is finite. Explicitly, the centralizer of $G$ in $\mathrm{GL}_n(\mathbb{Z})$ is isomorphic to the group of units in some order of the semi-simple algebra $\mathbb{Q}[G]$, and its rank is that of the group of units $U_n \simeq \prod_{d|n} E_d$ of the maximal order, where $E_d$ denotes the group of units of the $d$th cyclotomic field. As the rank of $E_n$ is positive except for $n \le 4$ and $n = 6$, the part of (1) concerning these values of $n$ is done. For the exceptional values of $n$, the group $\mathrm{SL}_G(\mathbb{Z})$ is isomorphic to the group $\mu_n$ of the $n$th roots of unity, it follows that it has order $n$, which completes the proof of (1) and transforms the obvious inclusions of $G$ (for $n=3$) or $G^+$ (for even $n$'s) in $\mathrm{SL}_G(\mathbb{Z})$ into equalities. For $n=5$, $\mathrm{SL}_G(\mathbb{Z})$ is isomorphic to $\mu_5\times\mathbb{Z}$, and $\sigma$ corresponds to the unit $\frac{1-\sqrt{5}}{2}$ in $\mathbb{Q}(\sqrt{5})$. To prove (3), we use obvious inclusions and (for $n=5$) a finiteness argument.

*Remark.* For $n=2,3,4,6$, the group $\mathrm{SL}_G(\mathbb{Z})$ acts trivially on the $G$-matrices (in other words $G$-integral equivalence is plain equality).

The rest of the paper is devoted to the complete explicit enumeration of $G$-cells up to dimension 5. We made use of the previous classifications of the whole set of matrices ([BM3], [B]). To find whether the regular representation (up to integral conjugacy) acts on a given class $c$, we used a rational matrix $E\in c$ fixed by the stabilizer $\mathrm{Aut}\,c$ of the class. Such a matrix can be constructed from any rational matrix $A\in c$ by the usual averaging formula:

$$E=\frac{1}{\mathrm{Aut}\,c}\sum_{g\in\mathrm{Aut}\,c}{}^t gAg,$$

(see [B], prop. 2.4). We then proceeded in the following way, that we explain for $n=4$ (the most tedious case). Let $\det E$ and $m$ be the determinant and minimum of $E$ rescaled so as to be integral. A circulant matrix with first row $(a,b,c,b)$ has determinant $(a-c)^2(a+2b+c)(a-2b+c)$, where $a$, $2(a-c)$, $4(a+2b+c)$ et $4(a-2b+c)$ are values of the quadratic form on integral vectors. In order to identify this matrix with $E$, we have to solve in integers the equation $(a-c)^2(a+2b+c)(a-2b+c)=\det E$, with $a\geq m$, $a-c\geq\frac{m}{2}$ and $a\pm 2b+c\geq\frac{m}{4}$. And eventually we test the integral equivalence of $E$ with the invariant matrix associated to each solution $(a,b,c)$. The computations were carried out with the PARI system.

## 4. Tables

In the following we give, for $n\leq 5$, the full lists of bounded $G$-cells up to equivalence, and the checks on Bavard's formula. In the tables, there is one line for each cell $c$, displaying its dimension $d=\dim c$, its eutactic representative $E$ (as it turned out that they all possess one), described by the first $1+\lfloor\frac{n}{2}\rfloor$ entries of its top row; we sometimes name the corresponding lattice $L$ (up to rescalation), making use of the usual notation ($\mathbb{A}_n,\mathbb{D}_n$ for the root lattices, $\mathbb{A}_n^r$ for the Coxeter lattices, $L^*$ for the dual lattice of $L$). We also indicate the number $s$ of pairs $\pm\boldsymbol{x}$ of minimal vectors, and the number $o$ of $G$-orbits of them. In the second column we indicate the "weight" $w$ of the cell in the mass formula, namely the number of matrices integrally equivalent to $E$ which are not $G$-integrally equivalent.

- $n = 2$

| $d$ | $w$ | $E$ | $L$ | $s$ | $o$ |
|---|---|---|---|---|---|
| 0 | 2 | $(1, \frac{-1}{2})$ | $\mathbb{A}_2$ | 3 | 2 |
| 1 | 1 | $(1, 0)$ | $(\mathbb{A}_1)^2$ | 2 | 1 |

$$\text{Mass formula :} \quad (2-1)\frac{1}{2} = \frac{1}{2}.$$

- $n = 3$

| $d$ | $w$ | $E$ | $L$ | $s$ | $o$ |
|---|---|---|---|---|---|
| 0 | 1 | $(1, \frac{1}{2})$ | $\mathbb{A}_3$ | 6 | 2 |
| 0 | 1 | $(1, \frac{-1}{3})$ | $\mathbb{A}_3^*$ | 4 | 2 |
| 1 | 1 | $(1, 0)$ | $(\mathbb{A}_1)^3$ | 3 | 1 |

$$\text{Mass formula :} \quad (1+1-1)\frac{1}{3} = \frac{1}{3}.$$

- $n = 4$

| $d$ | $w$ | $E$ | $L$ | $s$ | $o$ |
|---|---|---|---|---|---|
| 0 | 2 | $(1, \frac{1}{2}, \frac{1}{2})$ | $\mathbb{A}_4$ | 10 | 3 |
| 0 | 2 | $(1, \frac{1}{2}, \frac{1}{4})$ | $L_4^2$ | 9 | 3 |
| 0 | 2 | $(1, \frac{1}{8}, \frac{-1}{2})$ | | 7 | 3 |
| 1 | 2 | $(1, \frac{1}{2}, \frac{\sqrt{3}-1}{2})$ | | 8 | 2 |
| 1 | 2 | $(1, 0, \frac{1}{2})$ | $(\mathbb{A}_2)^2$ | 6 | 2 |
| 1 | 2 | $(1, \frac{-1}{4}, \frac{-1}{4})$ | $\mathbb{A}_4^*$ | 5 | 2 |
| 2 | 1 | $(1, 0, 0)$ | $(\mathbb{A}_1)^4$ | 4 | 1 |

$$\text{Mass formula:} \quad (6-6+1)\frac{1}{4} = \frac{1}{4}.$$

• $n = 5$

| $d$ | $w$ | $E$ | $L$ | $s$ | $o$ |
|---|---|---|---|---|---|
| 0 | 1 | $(1, 0, \frac{1}{2})$ | $\mathbb{D}_5$ | 20 | 4 |
| 0 | 1 | $(1, \frac{-1}{2}, \frac{1}{4})$ | $\mathbb{A}_5^3$ | 15 | 3 |
| 0 | 1 | $(1, \frac{1}{2}, \frac{1}{2})$ | $\mathbb{A}_5$ | 15 | 3 |
| 0 | 2 | $(1, \frac{-1}{2}, \frac{1}{10})$ | | 11 | 3 |
| 1 | 2 | $(1, \frac{1}{2}, \frac{\sqrt{5}-1}{4})$ | | 10 | 2 |
| 1 | 2 | $(1, \frac{-1}{2}, \frac{\sqrt{13}-3}{4})$ | | 10 | 2 |
| 1 | 1 | $(1, \frac{1}{3}, \frac{-1}{3})$ | $\mathbb{A}_5^2$ | 10 | 2 |
| 1 | 1 | $(1, \frac{-1}{5}, \frac{-1}{5})$ | $\mathbb{A}_5^*$ | 6 | 2 |
| 1 | 1 | $(\frac{13}{10}, \frac{-4}{5}, \frac{1}{4})$ | | 6 | 2 |
| 2 | 1 | $(\frac{5}{4}, \frac{1}{4}, \frac{-3}{4})$ | $\mathbb{D}_5^*$ | 5 | 1 |
| 2 | 1 | $(1, 0, 0)$ | $(\mathbb{A}_1)^5$ | 5 | 1 |

$$\text{Mass formula:} \quad (5 - 4 - 3 + 2)\frac{1}{5} = 0.$$

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