A GENERALIZATION OF SOME LATTICES OF COXETER

ANNE-MARIE BERGÉ AND JACQUES MARTINET

Abstract. This paper introduces a wide generalization of a family of integral lattices defined by Coxeter, which share with the Coxeter lattices the following properties: they are perfect, with often an odd minimum, and have no non-trivial perfect sections with the same minimum.

Let E be an n-dimensional Euclidean space with scalar product $x \cdot y$ and *norm* $N(x) = x \cdot x$. A *lattice in* E is a discrete subgroup Λ of E of rank n. We set $m = \min \Lambda = \min N(x)$, $x \in \Lambda \smallsetminus \{0\}$ (the *minimum of* Λ), $S = S(\Lambda) = \{x \in \Lambda \mid N(x) = m\}$ (the *sphere of* Λ), and $s = \frac{1}{2}|S|$ (its (*half*) *kissing number*).

In his 1951 paper [Cox], Coxeter considered the lattices L defined for odd $n \geq 5$ by the conditions $\mathbb{A}_n \subset L \subset \mathbb{A}_n^*$ and $[\mathbb{A}_n^* : L] = 2$, where $\mathbb{A}_n$ stands as usual for the root lattice generated by the root system of type A_n. He determined their minimum and proved that they are perfect, with $s = \frac{n(n+1)}{2}$, the smallest possible kissing number for a perfect lattice. In the sequel, we denote by Cox_n their scaled copy to the smallest minimum m which makes them integral ($2(n-1)$ if $n \equiv 1 \mod 4$, $\frac{n-1}{2}$ if $n \equiv 3 \mod 4$; see [M], Section 5.2).

In her paper [Be], the first author proved that these lattices are *hollow* for all $n \geq 7$ in the sense that they do not have any perfect r-dimensional sections with the same minimum in the range $1 < r < n$. (Cox_5, which has hexagonal sections of minimum m, is an exception.) Her method relies on the fact that $S(\mathbb{A}_n^*)$ is of the form $\{\pm v_0, \pm v_1, \dots, \pm v_n\}$ with vectors v_i which add to zero, and that the minimal vectors of Cox_n (up to sign) are then the sums $v_i + v_j$ for $0 \leq i < j \leq n$.

In relation with his recent joint paper [M-V] with Boris Venkov, the second author tried to construct in various dimensions integral perfect lattices Λ having an odd minimum, using a construction going back to Watson ([W]; see also [M1]): one starts with a lattice Λ' having a basis $(e_1, \dots, e_n)$ of minimal vectors, and then considers lattices of the form $\Lambda = \langle \Lambda', e\rangle$ for a vector $e = \frac{e'}{d}$, $e' \in \Lambda'$, $d \geq 2$. chosen in such a way that the minimal vectors of Λ lie in the set $S_0 = \{\pm e_i\} \cup \{\pm(e - e_i - e_j)\}$. It appeared that S_0 could be viewed as a set $\{\pm(v_i + v_j)\}$ where the v_i, $i = 0, 1, \dots, n$ belong to a lattice M containing Λ to index 2 and are related by a single relation $c_0 v_0 + c_1 v_1 + \cdots + c_n v_n = 0$ with at most one null coefficient c_i, an observation which throws new light on Watson's constructions. Using this device, we are able to prove:

Theorem 0.1. *For every $n \geq 10$, there exist perfect, hollow n-dimensional lattices having an odd minimum.*

The mere existence of perfect lattices having an odd minimum (hollow or not) for all large enough dimensions is new (the Coxeter family solves the problem only for dimensions congruent to 3 modulo 4). The proof of Theorem 0.1 will be given in Section 3. That these lattices are actually hollow lattices is proved following closely the arguments of [Be].

To guess systems (c_i) which ensure that all vectors in S_0 have the same odd norm is rather easy. As usual, the difficult problem is to show that these vectors are minimal. The theorem below (chosen because the corresponding lattices are highly symmetric; their automorphism group contains the symmetric group on n letters) is an instance of such a result. To state it, we introduce some more notation. We consider positive integers $a_1, \dots, a_n, d$ and a lattice $\Lambda = \langle \Lambda', e \rangle$ for which $e = (a_1 e_1 + \cdots + a_n e_n)/d$. We shall prove that there is a unique system of scalar products $(e_i \cdot e_j)$ such that all vectors in S_0 have a given common norm $m > 0$. Set $A = a_1 + \cdots + a_n$ and $q = A - 2d$. For the system (v_i), we then have $c_0 = q$ and $c_i = a_i$ if $i \geq 1$, and thus $C := \sum_{i=0}^{n} c_i = 2A - 2d$ is even, as well as $A - q$, and the transition between Coxeter-like and Watson-like constructions is given by the formulae $e_i = v_0 + v_i$ and $e = 2v_0$, whence $e - e_i - e_j = -(v_i + v_j)$, and in the other direction, $v_0 = \frac{e}{2}$ and $v_i = -(\frac{e}{2} - e_i)$.

Theorem 0.2. *If $n \geq 10$ and $a_1 = \cdots = a_n = 1$, the vectors of S_0 are minimal if and only if $0 \leq q \leq \sqrt{n+1} - 1$, and $S(\Lambda)$ reduces to S_0 if and only if these inequalities are strict. $\big($If $q = 0$, $S(\Lambda) = S_0 \cup \{\pm(e - e_i)\}$; if $q = \sqrt{n+1} - 1$, $S(\Lambda) = S_0 \cup \{\pm(e_i - e_j)\}$.$\big)$*

The proof will be given in Section 2. This suffices to prove Theorem 0.1 for all dimensions $n \geq 21$ not divisible by 8 (and indeed for all n outside a set of density zero). The missing dimensions will be dealt with using systems (a_i) with $a_1 = \cdots = a_{n-2} = 1$, $a_{n-1} = 1$ or 2, and $a_n = 2$, see Section 3 and the appendix.

1. A perfect system

In this section, we assume that E has dimension $n \geq 5$. We denote by $\mathrm{End}^s(E)$ the space of symmetric endomorphisms of E, and for any $x \in E$, by $p_x \in \mathrm{End}$ the orthogonal projection onto the line spanned by x (and set $p_0 = 0$). The *perfection rank* of a subset S of E is the rank of the system $\{p_x,\, x \in S\}$, and we say that S is *perfect* if this perfection rank is equal to $\dim \mathrm{End}^s(E) = \frac{n(n+1)}{2}$. As usual, a lattice Λ in E is *perfect* if the set of its minimal vectors is perfect.

We now give a method of construction of perfect lattices, starting from a system of $n+1$ vectors of E.

Theorem 1.1. *Let $\mathcal{V} = \{v_i,\, 0 \leq i \leq n\} \subset E$ be a set of $n+1$ distinct, non-zero vectors of E, of rank n, satisfying a non-trivial relation with integral coefficients. If the set*

$$S_{\mathcal{V}} = \{v_i + v_j, 0 \leq i < j \leq n\} \subset E$$

consists of minimal vectors of the lattice $\Lambda_{\mathcal{V}} = \langle S_{\mathcal{V}} \rangle$ it spans, then the set $S_{\mathcal{V}}$ and the lattice $\Lambda_{\mathcal{V}}$ are perfect.

We first establish a more general property of perfection for the set $S_{\mathcal{V}}$.

Lemma 1.2. *Let $\sum_{0 \leq i \leq n} c_i\, v_i = 0,\ c_i \in \mathbb{R}$ be a non-trivial relation between the $v_i \in \mathcal{V}$. Set $C = \sum_{0 \leq i \leq n} c_i$. Then, if the coefficients c_i satisfy the hypotheses*

$$c_i \neq \frac{C}{4} \text{ for } 0 \leq i \leq n \quad \text{and} \quad \sum_{0 \leq i \leq n} \frac{C}{C - 4c_i} - (n-3) \neq 0\,,$$

the set $S_{\mathcal{V}}$ is perfect. This is true in particular when $c_i < \frac{C}{4}$ for all i, which implies

$$\sum_{0\le i\le n} \frac{C}{C-4c_i} - (n-3) > 8\,.$$

Proof. Since $|\{(i,j), 0\le i<j\le n\}| = \frac{n(n+1)}{2}$, it suffices to prove that the $p_x, x\in S_{\mathcal{V}}$ are linearly independent in $\mathrm{End}^s(E)$. Consider in $\mathrm{End}^s(E)$ an endomorphism

$$\sum_{0\le i<j\le n} N(v_i+v_j)\,\lambda_{ij}\,p_{v_i+v_j} \quad\text{with}\quad (\lambda_{ij})\in\mathbb{R}^{\frac{n(n+1)}{2}}\,.$$

We suppose for example $c_0\neq 0$, and use the basis $\mathcal{B}=(v_1,\dots,v_n)$ and its dual basis $\mathcal{B}^*$. For convenience, we set

$$\forall\, i\ge 1,\,\forall\, j<i,\;\lambda_{ij}=\lambda_{ji},\;\mu_i=\frac{\lambda_{0i}}{c_0^2}\quad\text{and}\quad \mu=\sum_{1\le i\le n}\mu_i\,.$$

The $n\times n$ matrix (a_{hk}) of the endomorphism $\sum\lambda_{ij}N(v_i+v_j)\,p_{v_i+v_j}$ in the bases $(\mathcal{B}^*,\mathcal{B})$ is given by

$$a_{hk}=\begin{cases}\lambda_{hk}+c_hc_k\mu-c_0(c_k\mu_h+c_h\mu_k) & \text{if } h\neq k\\ \displaystyle\sum_{l\neq k}\lambda_{lk}+c_k^2\,\mu-c_0(2c_k-c_0)\mu_k & \text{if } h=k.\end{cases}\qquad(\mathbf{1.1})$$

This implies that $\sum_{h\neq k}a_{hk}-a_{kk}=c_k(C-2c_0-2c_k)\,\mu-c_0(C-4c_k)\,\mu_k$ for all $k\ge 1$. Using now the hypothesis $C-4c_k\neq 0$ for all k, we obtain

$$\frac{\sum_{h\neq k}a_{hk}-a_{kk}}{C-4c_k}=\frac{c_k(C-2c_0-2c_k)}{C-4c_k}\,\mu-c_0\,\mu_k\,,\qquad(\mathbf{1.2})$$

and thus

$$\sum_{1\le k\le n}\frac{\sum_{h\neq k}a_{hk}-a_{kk}}{C-4c_k}=\Big(\sum_{1\le k\le n}\frac{c_k(C-2c_0-2c_k)}{C-4c_k}-c_0\Big)\,\mu\,,\qquad(\mathbf{1.3})$$

where

$$\sum_{1\le k\le n}\frac{c_k(C-2c_0-2c_k)}{C-4c_k}-c_0=\frac{C-4c_0}{8}\Big(\sum_{0\le i\le n}\frac{C}{C-4c_i}-(n-3)\Big)\neq 0\,.$$

Now, suppose that $\sum N(v_i+v_j)\,\lambda_{ij}\,p_{v_i+v_j}$ is zero, i.e. that $a_{hk}=0$ for all h,k. It follows from (1.3) that $\mu=0$, hence by (1.2) that $\lambda_{0k}=\mu_k=0$ for all $k\ge 1$, and finally by (1.1) that $\lambda_{hk}=0$ for all $1\le h<k\le n$. Hence $S_{\mathcal{V}}$ is perfect.

Suppose now that $C-4c_i>0$ for all $i\ge 0$, which implies $C>0$, and thus $u_i=\dfrac{C}{C-4c_i}>0$. We have $\sum\dfrac{1}{u_i}=n-3$, hence

$$\sum_{0\le i\le n}\frac{C}{C-4c_i}-(n-3)=\sum_{0\le i\le n}u_i-\sum_{0\le i\le n}\frac{1}{u_i}\,.$$

Multiplying side by side the arithmetic-geometric inequalities

$$\sum_i u_i\ge(n+1)(\prod_i u_i)^{\frac{1}{n+1}}\qquad\text{and}\qquad\sum_i\frac{1}{u_i}\ge(n+1)(\prod_i\frac{1}{u_i})^{\frac{1}{n+1}}\,,$$

we obtain $\Big(\sum_i u_i\Big)\Big(\sum_j \frac{1}{u_j}\Big) \ge (n+1)^2$, where equality holds if and only if all u_i are equal. In particular,

$$\sum u_i - \sum \frac{1}{u_i} \ge \frac{(n+1)^2 - (\sum \frac{1}{u_i})^2}{\sum \frac{1}{u_i}}\,,$$

i.e. $\sum_{0\le i\le n} \frac{C}{C-4c_i} - (n-3) \ge \frac{8(n-1)}{n-3} > 8$. □

Proof of Theorem 1.1. We prove that the coefficients of the relation $\sum_{0\le i\le n} c_i v_i = 0$, where without loss of generality we may suppose $C = \sum c_i \ge 0$, satisfy the hypotheses of Lemma 1.2, under the mere assumptions

$$\begin{cases} \text{for } 0 \le i \le n : N(2v_i) \ge m \\ \text{for } 0 \le i < j \le n : N(v_i - v_j) \ge N(v_i + v_j) = m > 0\,. \end{cases} \qquad \mathbf{(1.4)}$$

[Indeed the lattice $\Lambda_{\mathcal{V}}$ contains the non-zero vectors $2v_i = (v_i + v_j) + (v_i + v_k) - (v_j + v_k)$ and $v_i - v_j = v_i + v_j - 2v_j$, and the hypothesis $\min \Lambda_{\mathcal{V}} = m$ implies conditions (1.4).]

For $i = 0, \dots, n$, let

$$\alpha_i = v_i \cdot v_i\,.$$

It results from the hypotheses (1.4) that for $i \ne j$, we have

$$\alpha_i + \alpha_j - m = -2v_i \cdot v_j = \frac{N(v_i - v_j) - N(v_i + v_j)}{2} \ge 0\,. \qquad \mathbf{(1.5)}$$

We now use the relation $\sum_{j\ne i} c_j(v_i \cdot v_j) = v_i \cdot \sum_{j\ne i} c_j v_j = -c_i\alpha_i$. By performing linear combinations on the equalities in (1.5), we obtain the $n+1$ Watson-like identities

$$\sum_{j\ne i} c_j \underbrace{\Big(N(v_i - v_j) - N(v_i + v_j)\Big)}_{\ge 0} = c_i\, N(2v_i)\,, \qquad (\mathbf{W}(\mathrm{i}))$$

and the relations $\sum_{j\ne i}(\alpha_i - m) + \sum_{j\ne i} c_j\alpha_j = 2\alpha_i c_i$, which also read

$$(C - 4c_i)\Big(\alpha_i - \frac{m}{4}\Big) = K \ \text{ where } \ K = \frac{m\,C}{2} - \sum_j c_j\Big(\alpha_j - \frac{m}{4}\Big)\,. \qquad \mathbf{(1.6)}$$

We now prove that K is strictly positive. Since $2v_i \in \Lambda_{\mathcal{V}}$ is non-zero and $4\alpha_i = N(2v_i)$, we have $\alpha_i \ge \frac{m}{4}$ for all i, and the inequalities in (1.5), namely $\alpha_i + \alpha_j \ge m$ for all $i \ne j$, show that there exists at most one index i_0 such that $\alpha_{i_0} \le \frac{m}{4}$, and then $\alpha_{i_0} = \frac{m}{4}$ by condition (1.4). Thus $K \le 0$ would imply by (1.6) $c_i \ge \frac{C}{4} \ge 0$ for all $i \ne i_0$, and Watson's identity ($\mathrm{W}(i_0)$) then implies $c_{i_0} \ge 0$. Since the c_i are non-negative and not all zero, their sum C is strictly positive and satisfies the inequality $C = \sum_{i\ne i_0} c_i + c_{i_0} \ge n\,\frac{C}{4}$, a contradiction for $n \ge 5$. From (1.6) and $K > 0$, it follows that $C > 4c_i$ (and $\alpha_i > \frac{m}{4}$) for all i, and then, by Lemma 1.2 the set $S_{\mathcal{V}}$ is perfect, and so is the lattice $\Lambda_{\mathcal{V}}$ if m is its minimal norm. (Actually, this last condition is fulfilled under the hypotheses (1.4) at least in the cases studied in Section 2.) This completes the proof of the theorem. □

From Formula (1.6) we can now make explicit K and the Gram matrix for the set $\mathcal{V}$. For further use, we state these results as a lemma.

Lemma 1.3. *Under the hypotheses of Theorem 1.1, one has*

$$K = \frac{2m\,C}{\sum_i \frac{C}{C-4c_i}} - (n-3)\,,$$

and

$$\forall\, i,\, j,\;\; v_i\cdot v_j = \begin{cases} \dfrac{m}{4} + \dfrac{K}{C-4c_i}\;(=\alpha_i) & \textit{if } j=i,\\ \dfrac{m-\alpha_i-\alpha_j}{2} & \textit{if } j\neq i. \end{cases}$$

[As above, $\sum_{i=0}^n c_i v_i = 0$ and $C=\sum_{i=0}^n c_i$.]

Proof. By (1.6) we have $\alpha_i - \frac{m}{4} = \frac{K}{C-4c_i}$ for all i, hence $K = \frac{m\,C}{2} - K\sum_i \frac{c_i}{C-4c_i}$, where $\sum_i \frac{c_i}{C-4c_i} = \frac{1}{4}\big(\sum_i \frac{C}{C-4c_i} - (n+1)\big)$; this shows that $a\,K = 2m\,C$ where $a = \sum_i \frac{C}{C-4c_i} - (n-3)$ is > 8 by Lemma 1.2. This gives K and the α_i, and the other scalar products stem from Formula (1.5). □

Incidentally, we have proved the upper bound $K < \frac{mC}{4}$, from which one can derive some other properties of the c_i, namely:
(i) $\forall\, i,\; c_i \geq 0$, and $c_i = 0$ for at most one index i_0 (note that from (W(i_0)), $c_{i_0} = 0 \Leftrightarrow N(v_i - v_{i_0}) = m$ for all $i \neq i_0$).
(ii) $C = \sum c_i$ is even; see the introduction.

2. Calculation of minima

In this section, we prove Theorem 0.2 and then sketch the proof of an analogous result that we shall also need in the next section to prove Theorem 0.1. The basic idea is to use a "decomposition into many squares" of the quadratic form $N(x)$.

Proof of Theorem 0.2. The notation is that of Lemma 1.2, with $c_0 = q$ and $c_i = 1$ for $i \geq 1$, where $0 \leq q \leq \sqrt{n+1} - 1$. Thus $C = n + q$ satisfies the conditions $C > 4c_i$. Lemma 1.3 shows that the scalar products $v_i \cdot v_j$ take at most four values, namely (with $1 \leq i < j \leq n$)

$$\alpha_0 = v_0\cdot v_0,\quad \alpha = v_i\cdot v_i,\quad \beta_0 = -v_0\cdot v_i,\quad \beta = -v_i\cdot v_j\,,$$

uniquely determined by the value of $m = N(v_i + v_j)$. For convenience, we give m the value

$$m = \frac{n^2 - 2n(q+1) + 4q - q^2}{2}\,,$$

a strictly positive (and even) integer (note that $A = n \equiv q \bmod 2$). We then have

$$\alpha_0 = \frac{1}{4}(n^2-3n),\quad \alpha = \frac{1}{4}\big(n^2 - n(2q+1) + 2q - 2q^2\big),$$
$$\beta_0 = \frac{1}{4}(n-3)q,\qquad \beta = \frac{1}{4}(n-2q-q^2)\,,$$

and the bounds we assumed for q are equivalent to conditions (1.5): for all $i \neq j$, $v_i\cdot v_j \leq 0 \iff N(v_i - v_j) \geq m$.

We first suppose $q \geq 1$, and use the basis $\mathcal{B} = (v_1,\dots,v_n)$ for E. The vector $x = \sum_{1\leq i\leq n} x_i v_i$ belongs to the lattice $\Lambda_{\mathcal{V}} = \langle v_i + v_j\rangle$ if and only if its components have the form $x_i = \frac{a_i}{q}$ where the integers a_i are pairwise congruent modulo q and have an even sum. Its norm is

$$N(x) = \big(\alpha - (n-1)\beta\big)\sum_i x_i^2 + \beta\sum_{i<j}(x_i - x_j)^2,$$

where $\alpha-(n-1)\beta = \frac{1}{4}\,(n-3)q^2$. Note that $m = 2\alpha-2\beta = 2(\alpha-(n-1)\beta)-2(n-2)\beta$, and hence

$$N(x) - m = \big(\alpha - (n-1)\beta\big)\big(Q_1 - 2\big) + \beta\big(Q_2 - 2(n-2)\big)\,, \qquad (\mathbf{2.1})$$

where $Q_1 = \sum_i x_i^2$ and $Q_2 = \sum_{i<j}(x_i - x_j)^2$. We consider three cases according to whether $\max|x_i - x_j|$ is 0, 1 or ≥ 2.

(i) *One value case.* Then $x = -a(v_1 + \cdots + v_n)/q = -av_0$, $a \in \mathbb{Z}, a \ne 0$, lies in the lattice if and only if a is even, which implies $N(x) \ge 4\alpha_0 > m$.

(ii) *Two values case*, say, $x_1 = \cdots = x_k$, $x_{k+1} = \cdots = x_n = x_1+1$. We then have $Q_2 = k(n-k) \ge 2(n-2)$ except for $k = 1$ or $n-1$, and $Q_1 = nx_1^2+(n-k)(2x_1+1) \ge k(n-k)/n$. If $k(n-k) \ge 3(n-3)$, then $Q_1 > 2$ (because $n \ge 10$), hence $N(x) > m$ by (2.1). It remains to discuss the cases $k(n-k) = (n-1)$ or $2(n-2)$.

Suppose first that $(x_i) \in \mathbb{Z}^n$. Then we have $Q_1 \ge 2$, with equality on the sequence $(1^2, 0^{n-2})$ (up to sign and permutations), for which $Q_2 = 2(n-2)$, $N(x) = m$, and x is one of the vectors $\pm(v_i + v_j)$, $1 \le i < j \le n$. When $Q_1 > 2$ and $k(n-k) \ge 2(n-2)$, we have $N(x) > m$ by (2.1). Next suppose that $k(n-k) = (n-1)$. We then have $Q_1 \ge n-1$, with equality attained on $(1^{n-1}, 0^1)$, and (2.1) shows that $N(x) - m \ge (n-3)(\alpha - n\beta)$, where $\alpha - n\beta = (q-1)(nq + n - 2q) \ge 0$. This implies $N(x) > m$ except when $q = 1$ and $Q_1 = n-1$, in which case we find the vectors $x = \pm(v_i + v_0)$ of a Coxeter lattice.

Consider now the case where $x_1 = \frac{a}{q}$, with $a \not\equiv 0 \mod q$ (which excludes the Coxeter lattices, for which $q = 1$). We then have $Q_1 = n(x_1 + 1 - \frac{k}{n})^2 + \frac{k(n-k)}{n}$, where we may assume (exchanging x_1 and $x_1 + 1$ if need be) that $k = 1$ or $k = 2$. From our hypotheses we have $\frac{q}{n} < \frac{1}{4}$; this shows that Q_1 has minimum

$$\frac{n - 2kq + kq^2}{q^2} = 2 + \frac{n - 2kq + (k-2)q^2}{q^2}\,,$$

attained only on $x_1 = \frac{1}{q} - 1$. For $k = 2$, we have $Q_1 > 2$, hence $N(x) > m$ by (2.1). For $k = 1$, formula (2.1) gives

$$\begin{aligned} N(x) - m &\ge \big(\alpha - (n-1)\beta\big)\,\frac{n - 2q - q^2}{q^2} - \big((n-3)\beta\big) \\ &= \frac{n - 2q - q^2}{q^2}\,(\alpha - (n-1)\beta - \frac{n-3}{q^2}) = 0\,, \end{aligned}$$

and $N(x) = m$ occurs only on $((\frac{1}{q} - 1)^1, \frac{1}{q}^{n-1})$, which corresponds to the vectors $\pm(v_0 + v_i)$.

(iii) *Case when there is a difference* $|x_i - x_j| \ge 2$ with, say, $x_1 \le x_2 \le \cdots \le x_n$. We have

$$Q_1 \ge x_1^2 + x_n^2 = \frac{(x_1 - x_n)^2 + (x_1 + x_n)^2}{2} \ge 2\,,$$

and equality holds only on the sequence $(-1^1, 0^{n-2}, 1^1)$.

Now from the identity $Q_2 = nQ_1 - (\sum x_i)^2$, it follows $Q_2 \ge 2n$: indeed Q_2 is invariant under translation, and we may assume $\sum x_i = 0$, and then $Q_2 = nQ_1 \ge$

$2n$. Formula (2.1) shows that $N(x) - m \geq 4\beta \geq 0$. Hence we have $N(x) > m$, except if $\beta = 0$ (i.e. $n = 2q + q^2$) and $Q_1 = 2$, i.e. for $x = \pm(v_i - v_j)$, $1 \leq i < j \leq n$.

So far, we left aside the case where $q = 0$, because $v_1, \ldots, v_n$ then no longer constitute a basis for E. To deal with this special case, we considered the less symmetric basis $(v_0, v_1, \ldots, v_{n-1})$. The theorem was proved using calculations analogous to the previous ones. Since we do not need this special case in the remaining of the paper, we leave the details to the reader. □

The theorem we have just proved does not suffice to prove Theorem 0.1. For this reason, we give below the analogous statement for systems $(a_i) = (1^{n-1}, 2)$, for which $A = n + 1$, hence $q = n + 1 - 2d$. The proof follows the same pattern, and we shall restrict ourselves to the case where $q = 3$ and $n \equiv 0 \mod 4$, which suffices for our applications.

Theorem 2.1. *If $n \geq 10$ and $(a_1, \ldots, a_n) = (1, \ldots, 1, 2)$, the vectors of S_0 are minimal if and only if $1 \leq q \leq \sqrt{n} - 1$, and S_0 is then the set of* all *minimal vectors of Λ.*
[Recall that $S_0 = \{\pm e_i\} \cup \{\pm(e - e_i - e_j)\} = \pm S_{\mathcal{V}}$.]

Proof (sketch for $q = 3$ and $n \equiv 0 \mod 4$). We work in the basis $(v_0, v_1, \ldots, v_{n-1})$ for E. Then $x = \sum_{i=0}^{n-1} x_i v_i$ belongs to Λ if and only if the x_i are either all integral or all halves of integers, and add to an even number. As above, for $1 \leq i < j \leq n-1$, we set $\alpha_0 = v_0 \cdot v_0$, $\alpha = v_i \cdot v_i$, $\beta_0 = -v_0 \cdot v_i$, and $\beta = -v_i \cdot v_j$, and give m the value

$$m = \frac{n^3 - 10n^2 + 24n - 16}{16}\,.$$

This gives the scalar products $v_i \cdot v_j$ the values

$$\alpha_0 = \frac{1}{32}(n^3 - 5n^2 + 4n - 8), \qquad \alpha = \frac{1}{32}(n^3 - 9n^2 + 4n + 56),$$
$$\beta_0 = \frac{1}{32}(3n^2 - 20n + 40), \qquad \beta = \frac{1}{32}(n^2 - 20n + 72)\,,$$

We write the norm in the form

$$\alpha_0 N(x) = (\alpha_0 x_0 - \beta_0 \sum_{i=1}^{n-1} x_i)^2 + B \sum_{i=1}^{n-1} x_i^2 + C \sum_{1 \leq i < j \leq n-1} (x_i - x_j)^2$$

where $B = \alpha_0(\alpha - (n-2)\beta) - (n-2)\beta_0^2$ and $C = (\alpha_0\beta + \beta_0^2)$. Using arguments similar to those we used in the proof of Theorem 0.2, we show that $\min \Lambda = m$, attained on the sequences (up to permutation and sign) $(\frac{1}{2}, \ldots, \frac{1}{2})$, $(\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and $(0, 1, 1, 0, \ldots, 0)$, which correspond to $v_0 + v_n$, $v_1 + v_n$ and $v_1 + v_2$ respectively. □

3. Hollow lattices having an odd minimum

In this section, we consider as in the previous sections lattices $\Lambda = \langle \Lambda', e \rangle$ with $e = \frac{a_1 e_1 + \cdots + a_n e_n}{d}$, such that $S(\Lambda) = S_0$.

Theorem 3.1. *A lattice of the form $\Lambda = \langle \Lambda', \frac{a_1 e_1 + \cdots + a_n e_n}{d} \rangle$ such that $S(\Lambda) = S_0$ and $\dim \Lambda \geq 7$ is hollow.*

Proof. We refer the reader to [Be], whose combinatorial methods used in the case where $c_i = 1$ for all i (that of the Coxeter lattices) extend in a straightforward way to non-necessarily equal strictly positive coefficients c_i. These methods indeed show that for any r-dimensional subspace F of E with $1 < r < n$, we have $|S_0 \cap F| \le \frac{r(r-1)}{2} + 1 < \frac{r(r+1)}{2}$ (and even $s(\Lambda \cap F) \le \frac{r(r-1)}{2}$ if $r \le n-2$). □

We now prove two lemmas on lattices Λ constructed with systems $(a_i) = (1^n)$ and $(1^{n-1}, 2)$ respectively, from which Theorem 0.1 will follow easily.

Lemma 3.2. *If one of the following three conditions holds:*

(1) $n \equiv 1 \mod 4$ *and* $n \ge 17$*;*
(2) $n \equiv 2 \mod 4$ *and* $n \ge 10$*;*
(3) $n \equiv 3 \mod 4$ *and* $n \ge 7$*,*

there exists an integral lattice of the form $\Lambda = \langle \Lambda', \dfrac{e_1 + \cdots + e_n}{d} \rangle$ *with an odd minimum and* $S(\Lambda) = S_0$.

Proof. We use the notation we introduced in the proof of Theorem 0.2 and start with the scale defined there, for which the vectors of S_0 have norm

$$m = \frac{n^2 - 2n(q+1) + 4q - q^2}{2}\,.$$

(Here, $A = n$ hence $q = n - 2d$.) We shall first choose $q \equiv n \mod 2$ in $[1, \sqrt{n+1} - 1)$ which ensures by Theorem 0.2 the existence of a lattice Λ with $\min \Lambda = m$ and $S(\Lambda) = S_0$, then divide out m by the highest power a of 2 it contains, and check that the scalar products on $a^{-1/2}\Lambda$ remain integers. Since Λ is generated by the vectors $e_i = v_0 + v_i$ and $e = 2v_0$, this last condition holds if and only if the three scalar products $(v_0 + v_1) \cdot (v_0 + v_2)$, $2v_0 \cdot 2v_0$ and $2v_0 \cdot (v_0 + v_i)$ are divisible by a. These are $\alpha_0 - 2\beta_0 - \beta$, $4\alpha_0$ and $2(\alpha_0 - \beta_0)$ respectively.

If $n \equiv 1$ (resp. 2) $\mod 4$, we take $q = 3$ (resp. $q = 2$), and $n \ge 17$ (resp. $n \ge 10$). In both cases, we have $m \equiv 2 \mod 4$. Theorem 0.2 ensures that $\min \Lambda = m$ and $S(\Lambda) = S_0$, and it is easily checked using the explicit values for α_0, β_0, β given in the proof of Theorem 0.2 that the three scalar products above are even.

If $n \equiv 3 \mod 4$, we take $q = 1$. Then Λ is proportional to Cox_n, which is known to be integral when scaled to minimum $\frac{n-1}{2}$; see [M], Proposition 2.5.3.

[If $n \equiv 2^k \mod 2^{k+1}$, there exists $n(k)$ such that some Λ can be rescaled to an integral lattice having an odd minimum if and only if $n \ge n(k)$, for instance, $n(4) = 28$, $n(8) = 88$. But dimensions in an infinite set of density zero are excluded, so that we cannot handle all large enough $n \equiv 0 \bmod 4$.] □

Lemma 3.3. *If* $n \equiv 0 \mod 4$ *and* $n \ge 16$*, there exists an integral lattice of the form* $\Lambda = \langle \Lambda', \dfrac{e_1 + \cdots + e_{n-1} + 2e_n}{d} \rangle$ *with an odd minimum and* $S(\Lambda) = S_0$.

Proof. We use the data given for $q = 3$ in the proof of Theorem 2.1. This time, it is easily verified that the value chosen there for m is odd, an that all scalar products $(v_0 + v_i) \cdot (v_0 + v_j)$, $2v_0 \cdot 2v_0$ and $2v_0 \cdot (v_0 + v_i)$ are integral (and incidentally odd). □

Proof of Theorem 0.1. If $n \ne 12$, 13, Theorem 0.1 is an immediate consequence of Lemma 3.3 if n is divisible by 4 and of Lemma 3.2 otherwise. If $n = 12$ or $n = 13$, we consider explicit Gram matrices relative to systems $(1^{n-2}, 2^2)$; see the appendix below. □

Appendix 1: Numerical data

We first list for each dimension $n \in [10, 25]$ one perfect, hollow lattice which can be used to prove Theorem 0.1, giving the minimum m, the index $d = [\Lambda : \Lambda']$, and the vector $\mathcal{A} = (a_1, \dots, a_n)$ in the usual symbolic form $(1^x, 2^y, \dots)$. We have considered vectors $(1^{n-k}, a_{k+1}, \dots, a_n)$ for small values of k and of the a_i, $i > k$. (When the sum $a_{k+1} + \cdots + a_n$ becomes too large, low-dimensional examples no more exist.) We have listed below the smallest minimum m we could find and then for given m, the smallest denominator d. For $n \equiv 3 \mod 4$, our lattice is Cox_n, for which $m = d = \frac{n-1}{2}$ and $a_1 = \cdots = a_n = 1$.

$n = 10$	$m = 11$	$d = 4$	(1^{10})	$n = 18$	$m = 11$	$d = 8$	(1^{18})
$n = 11$	$m = 5$	$d = 5$	(1^{11})	$n = 19$	$m = 9$	$d = 9$	(1^{19})
$n = 12$	$m = 31$	$d = 6$	$(1^{10}, 2^2)$	$n = 20$	$m = 91$	$d = 10$	$(1^{18}, 2^2)$
$n = 13$	$m = 15$	$d = 7$	$(1^{11}, 2^2)$	$n = 21$	$m = 15$	$d = 10$	$(1^{20}, 2)$
$n = 14$	$m = 29$	$d = 6$	(1^{14})	$n = 22$	$m = 89$	$d = 10$	(1^{22})
$n = 15$	$m = 7$	$d = 7$	(1^{15})	$n = 23$	$m = 11$	$d = 11$	(1^{23})
$n = 16$	$m = 57$	$d = 8$	$(1^{14}, 2^2)$	$n = 24$	$m = 19$	$d = 12$	$(1^{22}, 2^2)$
$n = 17$	$m = 39$	$d = 7$	(1^{17})	$n = 25$	$m = 107$	$d = 11$	(1^{25})

We list below experimental results for lattices with $A \le n+2$, giving an interval I such that $S(\Lambda) = S_0 \iff q \in I$ except for a few special values of n. The data below have been checked for all dimensions $n \le 100$. Of course, q must moreover satisfy the congruence $q \equiv A \mod 2$.

$\underline{A = n}$, $\mathcal{A} = (1^n)$: $1 \le q < \sqrt{n+1} - 1$; $n \ge 7$, $n \ne 8$.
$\underline{A = n+1}$, $\mathcal{A} = (1^{n-1}, 2)$: $1 \le q \le \sqrt{n} - 1$; $n \ge 10$.
$\underline{A = n+2}$, $\mathcal{A} = (1^{n-1}, 3)$: $1 \le q \le \sqrt{n-3} - 1$; $n \ge 16$, $n \ne 19$.
$\underline{A = n+2}$, $\mathcal{A} = (1^{n-2}, 2^2)$: $1 \le q \le \sqrt{n-1} - 1$; $n \ge 11$.

The minima we found are sometimes very large: 107 (see above) for $n = 25$, but also 241 for $n = 32$, 791 for $n = 65$, 1121 for $n = 70$, or 2257 for $n = 96$. It would be interesting to construct integral perfect lattices (maybe, non-hollow) having a much smaller minimum, and in particular minimum $m = 3$, for which our knowledge is very poor.

[Besides Cox_7, we know examples for $15 \le n \le 23$, using cross-sections of the unimodular lattice O_{23} listed in [Bt-M]). Using the classification of unimodular lattices of minimum 3 up to $n = 28$ (due to Bacher and Venkov in [Bc-V], extending previous works of Borcherds and of Conway), we see that for $n = 24$ to 28, perfect unimodular lattices of minimum 3 exist for $n = 27$ (1 lattice out of 3) and $n = 28$ (28 lattices out of 38). Finally, two perfect lattices (in dimensions 29 and 31) can be found in Nebe and Sloane's catalogue [N-S]. Note that it is proved in [M-V] that integral perfect lattices of minimum 3 do not exist in dimensions 8 and 9.]

Appendix 2: Extreme Lattices

Recall that a *eutaxy relation* among vectors of a finite set $T \subset E$ is an equality

$$\mathrm{Id} = \sum_{x \in T} \lambda_x \, p_x \, ,$$

that T is *eutactic* if there exists a eutaxy relation with strictly positive coefficients λ_x, and that a lattice Λ is *eutactic* if $S(\Lambda)$ is. Note that eutaxy relations exist if T is perfect, that such a relation is then unique if and only if $|T| = \frac{n(n+1)}{2}$, and that a lattice is *extreme* (i.e., the density its corresponding canonical sphere packing is a local maximum) if and only if it is both perfect and eutactic.

We have considered our set S_0 in the case where Λ is of the form

$$\langle \Lambda', \frac{e_1 + \cdots + e_n}{d} \rangle$$

and $S(\Lambda)$ contains S_0 (i.e., setting as usual $n = 2d + q$, when $0 \le q \le \sqrt{n+1} - 1$). Due to the action of the symmetric group S_n, the eutaxy relation has the form

$$\mathrm{Id} = \lambda \sum_{1 \le i < j \le n} p_{v_i + v_j} + \mu \sum_{1 \le k \le n} p_{v_0 + v_k} .$$

Calculating the trace and the image of v_0, we obtain

$$\lambda = \frac{n^2 - 2(q+2)n + q^2 + 8q}{(n-3)(n+q)(n-3q)} \text{ and } \mu = \frac{n^2 - 2(q^2+q+1)n + 5q^2 + 8q}{(n-3)(n+q)(n-3q)} ,$$

with which easily characterize the eutaxy property. Using Theorem 3.6.2 of [M] when $q = 0$ and a direct calculation when $q = \sqrt{n+1} - 1$, we have proved:

Theorem. *Let Λ be of the form $\langle \Lambda', \frac{e_1 + \cdots + e_n}{d} \rangle$, with $S(\Lambda) \supset S_0$. Then Λ is extreme if and only if $q \le \frac{\sqrt{2n+1}-1}{2}$ or $q = \sqrt{n+1} - 1$.* □

Note that λ and μ are equal if and only if $q = 1$. This proves that for $1 < q < \sqrt{n+1} - 1$, there are exactly two orbits of minimal vectors in Λ (those of $v_0 + v_1$ and of $v_1 + v_2$), then that $\mathrm{Aut}(\Lambda) \simeq \{\pm \mathrm{Id}\} \times S_n$. If $q = 1$, $\Lambda = \mathrm{Cox}_n$, there is a unique orbit, and $\mathrm{Aut}(\Lambda) \simeq \{\pm \mathrm{Id}\} \times S_{n+1}$. (If $q = 0$ or $q = \sqrt{n+1} - 1$, there are three orbits.)

Appendix 3: The Watson Identity

A *perfection relation* on a finite subset S of E is a dependence relation

$$\sum_{x \in S} a_x \, p_x = 0$$

in $\mathrm{End}^s(E)$ between the orthogonal projections p_x onto the vectors $x \in S$.

Given a lattice Λ written in the form $\Lambda = \langle \Lambda', e \rangle$ for a lattice Λ' having a basis $(e_1, \dots, e_n)$ and a vector

$$e = \frac{a_1 e_1 + \cdots + a_n e_n}{d} , \quad a_i, \, d \in \mathbb{Z} , \; d \ge 2 ,$$

Watson's identity is

$$\sum_{i=1}^{n} a_i \big(N(e - e_i) - N(e_i) \big) = (A - 2d) N(e) \qquad (*)$$

(or a slightly modified form involving the absolute values of the a_i). In the language of Section 1, there are $n+1$ "Watson's identities", one for each choice of a c_i, exactly one of which is equivalent to the identity displayed above.

With this identity is associated a unique perfection relation between the $2n + 1$ vectors e_i, $e'_i = e - e_i$, and e. Setting $m_i = N(e_i)$ and $m'_i = N(e'_i)$, and denoting by p_i, p'_i and p the projections onto e_i, e'_i and e respectively, it reads

$$\sum_{i=1}^{n} a_i \, (m_i \, p_i - m'_i \, p'_i) = -(A - 2d) \, N(e) \, p . \qquad (**)$$

The proof, whose details will be left to the reader, consists in writing a perfection relation of the type above with indeterminate coefficients, then evaluating both sides on two vectors e_i, e_j and on a basis of the space orthogonal to the span of e_i, e_j.

As a consequence, we obtain:

Proposition. *The perfection rank of the system* $\{p_i, p_i'\}$ *is equal to* $2n$ *if* $A \neq 2d$, *but only to* $2n-1$ *when "Watson's condition"* $A = 2d$ *holds.* □

In the applications, it is generally assumed that $e_1, \dots, e_n$ are minimal. Then $(*)$ shows that the vectors $e_i' = e - e_i$ also are minimal, and $(**)$ then shows that the perfection rank of Λ is strictly smaller that its kissing number s. We could show that in dimensions $n \leq 6$, all perfection relations stam from a relation of type $(**)$, with a denominator d equal to 2 or 3. This is no longer true if $n \geq 8$, where for instance perfection relations related to an identity of Zahareva (see[M1], proof of Proposition 9.1) occur. It would be interesting to deal with the case of dimension 7, where denominators $d = 2, 3, 4$ must be considered.

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A.-M. Bergé, J. Martinet
Inst. Math., Université Bordeaux 1
351, cours de la Libération
33405 Talence cedex, France
e-mail: berge@math.u-bordeaux.fr
martinet@math.u-bordeaux.fr

11H55: *NUMBER THEORY;*
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