ON LATTICES OF MAXIMAL INDEX TWO

ANNE-MARIE BERGÉ

ABSTRACT. The maximal index of a Euclidean lattice L of dimension n is the maximal index of the sublattices of L spanned by n independent minimal vectors of L. In this paper, we prove that a perfect lattice of maximal index two which is not provided by a cross-section has dimension at most 5.

1. Introduction

Korkine and Zolotareff proved that an n-dimensional lattice containing at least $\frac{n(n+1)}{2}$ pairs $\pm x$ of minimal vectors, and spanned by any subset of n independent minimal vectors, is similar to the root lattice $\mathbb{A}_n$.

Here we consider in an n-dimensional Euclidean space E well rounded lattices, i.e. lattices L the minimal vectors of which span E. To such a lattice L, Martinet attached some invariants related to the sublattices M of L generated by n independant minimal vectors of L, in particular the set of possible indices $[L : M]$, and for a given sublattice M, the group structure of the quotient L/M.

The *maximal index* of L is :

$$\max_M [L : M] ,$$

where M runs through sublattices of L spanned by n independent minimal vectors of L. (Korkine-Zolotareff's result deals with lattices of maximal index 1.)

In this paper, we consider lattices with maximal index 2. For such lattices, the notion of length introduced in [M] can be defined as follows:

The *length* $\ell \leq n$ *of a lattice* L *of maximal index* 2 is the minimal cardinality $|X|$ of a set X of independent minimal vectors of L such that $\sum_{x \in X} x \equiv 0 \mod 2L$.

Up to dimension 7, there are six perfect lattices with maximal index 2: in Conway-Sloane's notation (see [C-S] p. 56), P_4^1 and P_5^1 have length $\ell = 4$, while P_5^2, P_6^5, P_6^6 and P_7^{32} have length $\ell = 5$. In

Key words and phrases. Euclidean lattices, perfection
Université de Bordeaux, UMR 5251, Bordeaux, F-33000, France .

dimension 8, a computation by Batut and Martinet based on the classification result by Dutour-Schürmann-Vallentin (see [D-S-V]) showed that no 8-dimensional perfect lattice has maximal index 2.

In [M], Martinet conjectured that *a perfect lattice of maximal index* 2*, generated by its minimal vectors, has dimension at most* 7*.*

In the present work, we prove this conjecture in the case $\ell = n$.

Theorem 1.1. *A lattice of dimension $n \geq 6$, of maximal index* 2 *and length $\ell = n$, has less than $\frac{n(n+1)}{2}$ pairs $\pm x$ of minimal vectors, and in particular is not perfect.*

Actually, we shall obtain in 8.1 an asymptotic bound

$$s \leq \frac{2n^2}{9}$$

for the number s of pairs of minimal vectors much smaller than the (lower) perfection bound $\frac{n^2}{2}$.

2. Notation

Let L be a lattice of dimension $n \geq 6$, maximal index 2 and length n. Let $S = S(L)$ and $s(L) = \frac{|S(L)|}{2}$ denote the set and number of pairs $\pm x$ of minimal vectors of L.

Let $L_0 \subset L$ be a sublattice of index 2 generated by n independent minimal vectors $e_1, \dots e_n$ of L. We have $L = \langle L_0, e \rangle$, where, by possibly reducing e modulo L_0, and using the definition of the length, we may prescribe

$$e = \frac{e_1 + \cdots + e_n}{2}.$$

The hypotheses on the maximal index and the length of L imply that the minimal vectors of L_0 are just the $\pm e_i$, and that the other possible minimal vectors of L are of the form

$$\frac{\pm e_1 \pm e_2 \pm \cdots \pm e_n}{2}.$$

(See [M], Proposition 2.1.) In order to prove Theorem 1.1, we may and shall assume that $s(L) \geq n+1$, and in particular, by negating some e_i if necessary, *we shall suppose e itself minimal* (unless otherwise specified in Section 3). The next sections are devoted to the other minimal vectors $x \in S(L) \smallsetminus S(L_0)$, that we represent by their set I of minus signs:

$$x = x_I = e - \sum_{i \in I} e_i .$$

We call *type of* x the number $|I|$ of minus signs in the expression of x (e is of type 0). Of course the types of x and $-x$ add to n, therefore by possibly negating x we shall suppose $|I| \le \frac{n}{2}$, and if $|I| = \frac{n}{2}$ we shall prescribe $1 \in I$. [The index set I associated to the minimal vector x, and *a fortiori* its type, depend on the choice of $e \in L \smallsetminus L_0$.]

The following notation is relative to a given set of $r \ge 3$ minimal vectors $x_1, x_2, \ldots, x_r$ in $L \setminus L_0$ identified to their index sets $I_1, I_2, \ldots I_r$

$$x_k = x_{I_k} = e - \sum_{i \in I_k} e_i, \quad I_k \subsetneqq \{1, \ldots, n\}, \quad |I_k| \le \frac{n}{2}.$$

We denote by

$$m = |\cup_k I_k| \quad (m \le n)$$

the number of indices involved in the expression of the x_k. Actually, we may and shall suppose that

$$\bigcup_k I_k = \{1, 2, \ldots, m\}.$$

For $i = 1, \ldots, n$ we call *weight* of i the number $w(i) = 0, \ldots, r$ of subsets I_k it belongs to; we thus have

$$\sum_{k=1}^{r} x_k = r\, e - \sum_{i=1}^{n} w(i) e_i. \tag{1}$$

We also introduce the partition of $\cup I_k = \{1, \ldots, m\}$ into sets of indices of given weights

$$W_k = \{i \in \cup_k I_k \mid w(i) = k\} \quad (1 \le k \le r),$$

that we regroup into the sets of indices of even and odd weights

$$\mathcal{W}_0 = W_2 \cup W_4 \cup \ldots \quad \text{and} \quad \mathcal{W}_1 = W_1 \cup W_3 \ldots.$$

Section 3 gives properties about the weights in families of 3, 4 or 5 minimal vectors; these results are used in Sections 4 to 7 to give an upper bound for the *number t_p of minimal vectors of a given type p.*

[The bounds for t_1, t_2 and $t_1 + t_2$ given in Sections 3 and 4 were obtained by Martinet and the author while giving a classification of the six-dimensional perfect lattices based on their maximal index, work previously done by Baranovskii and Ryshkov in [B-R].]

Section 8 concludes by an estimation of the "kissing number" $s(L) = n + t_0 + t_1 + \cdots + t_{\lfloor \frac{n}{2} \rfloor}$ of L ($t_0 = 1$) strictly smaller than the dimension $\frac{n(n+1)}{2}$ of the space of lattices.

3. Properties of a set of minimal vectors

3.1. **Minimal vectors of type 1.** The following property derives from the hypothesis "*no n independent vectors of L span a sublattice of index* 3 *of L*" and does not suppose e minimal.

Proposition 3.1. *Suppose $n \geq 5$. Then there exist at most four minimal vectors of the form $e - e_i$ (i.e. $t_1 \leq 4$).*

Proof. Let $x_i = e - e_i$, $i = 1, \dots, 5$ be five minimal vectors of type 1 of L; using (1) we obtain

$$\sum_{i=1}^{5} x_i - \sum_{i=6}^{n} e_i = 5e - \sum_{i=1}^{n} e_i = 3e\,;$$

Clearly the n vectors $x_1, \dots, x_5, e_6, \dots, e_n$ are linearly independent, and generate a sublattice L' of index 3 in L, a contradiction. □

3.2. **Weights in a set of minimal vectors.** These properties of a set of $r = 3, 4$ or 5 minimal vectors of the form $x_k = e - \sum_{i \in I_k} e_i$ make essential use of the assumption that $\ell = n$, i.e. that *any set $X \subset S(L)$ of independent minimal vectors satisfying a congruence $\sum_{x \in X} x \equiv 0$* mod $2L$ *has cardinality $|X| = n$.* We first focus on the case $r = 4$, and here again e is not supposed to be minimal.

Lemma 3.2. *If every set I_1, I_2, I_3 and I_4 contains at least one index of weight* 1*, then this index is unique, and there is no index of weight* 3*.*

Proof . From (1) follows

$$\begin{aligned}\sum_{k=1}^{r} x_k + \sum_{i \in \mathcal{W}_0} e_i &= re - \sum_{i \in \mathcal{W}_1} w(i) e_i - \sum_{i \in \mathcal{W}_0} (w(i) - 1) e_i \\ &= 4e - \sum_{i \in W_1 \cup W_2} e_i - 3 \sum_{i \in W_3 \cup W_4} e_i \\ &= 4e - \sum_{i=1}^{m} e_i - 2 \sum_{i \in W_3 \cup W_4} e_i\,,\end{aligned}$$

where $\sum_{i=1}^{m} e_i = 2e - \sum_{i=m+1}^{n} e_i$, and thus we obtain

$$\sum_{k=1}^{4} x_k + \sum_{i \in \mathcal{W}_0} e_i - \sum_{i=m+1}^{n} e_i = 2e - 2x\,,$$

with $x = \sum_{i \in W_3 \cup W_4} e_i \in L$. Thus the set

$$X = \{x_1, x_2, x_3, x_4\} \cup \{e_i, i \in \mathcal{W}_0 \text{ or } i \geq m+1\}$$

of minimal vectors of L (which does not include the vector e) satisfies the congruence

$$\sum_{k=1}^{4} x_k + \sum_{i\in\mathcal{W}_0} e_i + \sum_{i=m+1}^{n} e_i \equiv 0 \mod 2L\,.$$

Its cardinality is

$$|X| = 4 + |\mathcal{W}_0| + (n-m) = n - (|W_1| - 4) - |W_3|$$

where $|W_1| \geq 4$ since for $k = 1, \dots, 4$, $W_1 \cap I_k \neq \emptyset$. To complete the proof of the lemma, it remains to prove that X is free. Suppose

$$\sum_{k=1}^{4} \lambda_k x_k + \sum_{i\in\mathcal{W}_0\cup\{m+1,\dots,n\}} \mu_i e_i = 0 \tag{2}$$

where the λ_k, μ_i are real numbers. Fix $k \in \{1, \dots, 4\}$; by assumption, there exists $i_k \in I_k$ of weight 1, hence belonging to no other I_h. With respect to the basis $e_1, \dots, e_n$ for E the coefficient a_{i_k} of the left hand side of (2) on the corresponding e_{i_k} reads $a_{i_k} = \frac{\sum_h \lambda_h}{2} - \lambda_k$. Its vanishing implies that the λ_k have a common value λ satisfying $2\lambda = \lambda$, hence $\lambda = 0$. Now (2) reduces to $\sum_{i\in\mathcal{W}_0\cup\{m+1,\dots,n\}} \mu_i e_i = 0$, and all μ_i are zero. The set X is free, which completes the proof. □

From now on, we suppose that e is a minimal vector of L.

Proposition 3.3. *If every I_k, $1 \leq k \leq r$, contains at least one index of weight 1, and if moreover when $r = 3$ there is an index of weight 3, then r is equal to 3 or 4, the index of weight 1 in every I_k is uniquely determined, and for $r = 3$ (resp. 4) we have $|W_3| = 1$ (resp. $|W_3| = |W_4| = 0$).*

Proof. The case $r \geq 5$ follows from the case $r = 4$ and Proposition 3.1.

(a) *Case* $r = 3$. By assumption, there exists an index of weight 3, say $1 \in I_1 \cap I_2 \cap I_3$. We change e_1 into $e_1' = -e_1$ and e into $e' = e - e_1 = \frac{e_1' + e_2 + \cdots + e_n}{2}$ (not necessarily minimal), and we consider the four minimal vectors $x_0 = e$, x_1, x_2 and x_3 which, relatively to e', read $x_k = e' - \sum_{i\in I_k'} e_i$ with $I_0' = \{1\}$, and $I_k' = I_k \setminus \{1\}$ for $k = 1, 2, 3$. The weights $w(i)$ and $w'(i)$ of an index i relative to the sets (I_1, I_2, I_3) and (I_0', I_1', I_3', I_4') coincide, except for $i = 1$: $w(1) = 3$ and $w'(1) = 1$. Thus the four minimal vectors x_i, $0 \leq i \leq 3$ satisfy the hypotheses of Lemma 3.2: there is no index $i \geq 2$ of weight 3, and the indices of weight 1 in I_1, I_2, I_3 are uniquely determined, as announced.

(b) *Case* $r = 4$. It remains to prove that $W_4 = \emptyset$. Otherwise, any subset of three I_k should satisfy the hypotheses of (a), hence $|W_4| =$

1, and by considering convenient ones we should obtain $|W_2| = 0$ (if $W_2 \cap I_1 \cap I_2 \neq \emptyset$, we consider the subset $\{I_2, I_3, I_4\}$, where I_2 has too many indices of weight one). Since by the lemma we already know that $|W_1| = 4$ and $|W_3| = 0$, every I_k should contain just one index of weight 1, say i_k, and one index of weight 4, say 1: the x_k are of the form $x_k = e - e_1 - e_{i_k}$, where the $i_k \geq 2$ are pairwise distinct. By the same substitution $e_1 \mapsto e'_1 = -e_1$, $e \mapsto e' = e - e_1$, (e' is not necessarily minimal), we obtain five vectors of type 1, namely $x_0 = e' - e'_1$ and the four $x_k = e' - e_{i_k}$, a contradiction with Proposition 3.1.

□

Application 3.4. • *Four pairwise disjoint sets I_k are singletons.*

• *Let $x_0 = e - \sum_{i \in I_0} e_i$ be a minimal vector of type $p = |I_0| \geq 3$, and let $A \subset I_0$, with $1 \leq |A| \leq p - 1$. There exists at most one vector x_I of type p such that $I \cap I_0 = A$.*

We now interchange the parts of even and odd weights, and focus on weight 2.

Proposition 3.5. *Let $x_1, \dots, x_r$ be $r \geq 3$ minimal vectors, of the form $x_k = e - \sum_{i \in I_k} e_i$. For $1 \leq k < k' \leq r$ we define the relation*

$$I_k \sim I_{k'} \iff I_k \cap I_{k'} \cap W_2 \neq \emptyset\,.$$

We suppose that the graph of the relation $\sim$ is a cycle of length $r = 3$ or 5, or a star of valency 3 (with $r = 4$).

Then the dimension n is equal to m or $m + 1$, where $m = |\bigcup_{k=1}^{r} I_k|$; moreover, if $n = m + 1$, then $|W_2| = r$ (resp. $r - 1 = 3$) in the case of a cycle (resp. star).

Proof. Note that the cycle (resp. star) contains r (resp. $r - 1$) edges, and thus that the number $|W_2|$ of indices of weight 2 is $\geq r$ (resp. $r - 1$). Since there is nothing specific to prove in the case $n = m$, we shall suppose $n \geq m + 1$ and show that then all equalities about n and $|W_2|$ hold.

We consider the set

$$X = \{x_1, \dots, x_r, e_i\ (i \in \mathcal{W}_1), \rho e\}$$

of minimal vectors, where $\rho \in \{0, 1\}$ is the remainder of r modulo 2, i.e. $\rho = 1$ in the case of the cycle, and 0 in the case of the star. Using (1) we obtain that the vectors of X add to a congruence modulo $2L$:

$$\sum_{k=1}^{r} x_k + \sum_{i \in \mathcal{W}_1} (w(i) - 2) e_i + (4 - r) e = \sum_{i \in \mathcal{W}_0} (2 - w(i)) e_i + 2 \sum_{i=m+1}^{n} e_i\,. \quad (4)$$

We now prove that the assumption $n \geq m+1$ implies that X is free. Let λ_k $(k=1,\dots,r)$, μ_i $(i \in \mathcal{W}_1)$, μ (equal to zero in the case of the star) be real numbers such that

$$\sum_{k=1}^{r} \lambda_k x_k + \mu e + \sum_{i \in \mathcal{W}_1} \mu_i e_i = 0 \,. \tag{5}$$

Put $a = \frac{\sum_{k=1}^{r} \lambda_k + \mu}{2}$. With respect to the basis (e_i) Condition (5) reads:

$$\begin{cases} a - \sum_{k \mid i \in I_k} \lambda_k = 0 & \forall i \in \mathcal{W}_0 \\ a - \sum_{k \mid i \in I_k} \lambda_k + \mu_i = 0 & \forall i \in \mathcal{W}_1 \\ a = 0 & \forall i \geq m+1. \end{cases} \tag{5'}$$

Since $n \geq m+1$, we can write Condition (5') for $i=n$, and we obtain $a=0$, i.e. $\sum \lambda_k = -\mu$.

Now, if $I_k \sim I_{k'}$ are adjacent, we obtain $\lambda_k = -\lambda_{k'}$ by writing Condition (5') for some $i \in W_2 \cap I_k \cap I_{k'}$. In the case of the 3-star with node I_1, it follows $\lambda_2 = \lambda_3 = \lambda_4 = -\lambda_1$, with $\sum \lambda_k = 0$ since $\mu = 0$, and thus $\lambda_k = 0$ for all k. In the case of the odd cycle say $(I_1, I_2, \dots, I_r)$, the λ_k takes the values λ_1 and $\lambda_2 = -\lambda_1$ alternatively; since r is odd, all λ_k vanish again, and so do $\sum \lambda_k$ and μ.

Eventually, in both cases (star or cycle), the conditions (5') give $\mu_i = 0$ for all $i \in \mathcal{W}_1$. Thus, when $n \geq m+1$, the set X is free. Since its vectors add to a congruence modulo $2L$, we must have $|X| = n$, where

$$\begin{aligned} n - |X| &= (|W_2| - r) + (n-m-1) \text{ in the case of the odd cycle} \\ &\quad (|W_2| - (r-1)) + |W_4| + (n-m-1) \text{ in the case of the star}\,. \end{aligned}$$

The terms between brackets in the right-hand sides are non-negative, and since $n - |X| = 0$ they vanish, as stated. □

4. Sets of minimal vectors of type at most two

The type 1 was dealt with in Proposition 3.1. We now focus on the type 2, i.e. on minimal vectors of the form $x = e - e_i - e_j$, $1 \leq i < j \leq n$.

Theorem 4.1. *We define on the set $\{1, 2, \dots, n\}$ the relation*

$$i \equiv j \quad \textit{if and only if} \quad e - e_i - e_j \quad \textit{is a minimal vector}.$$

Then, if $n \geq 6$, the graph of the relation $\equiv$ is a subgraph of the complete bipartite graph with 9 edges, except for $n = 6$ where it can also be a cycle of length 5.

Proof. We discard isolated vertices. By 3.3 we know that the valencies of the vertices are at most equal to 3, and that a disconnected graph contains no vertex of valency 3. By Proposition 3.5, the graph of the relation $\equiv$ contains no triangle (since $n > 4$) and no pentagon except for $n = 6$. If the graph is connected (resp. disconnected), it contains no path of length ≥ 6 (resp. ≥ 4) and no cycle of length ≥ 7 (resp. 5): otherwise, we could extract four minimal vectors whose graph has 3 connected components, a contradiction with 3.3. Now we conclude that every possible linear graph has at most 6 (non-isolated vertices) say $V = \{1, 2, 3, 4, 5, 6\}$, and that except the pentagon, they are bipartite: we can define a partition $V = V_0 \cup V_1$, $|V_0| = |V_1| = 3$ such that no two vertices in the same V_k are adjacent. It remains to consider a connected graph with at least one vertex of valency 3, say 1. Denote by $V_0 = \{2, 4, 6\}$ the set of its adjacent vertices. By Proposition 3.3, any other edge must be connected to this star, i.e. one of its end-points belongs to V_0, but not the other one (triangles are forbidden). Let V_1 denote the set vertices adjacent to vertices in V_0. It contains at most 3 vertices, as we shall now prove. If a vertex in V_0, say 2, has valency 3, exchanging the roles of the vertices 1 and 2, we see that V_1 is the set of the vertices adjacent to 2, and thus $|V_1| = 3$. If no vertex in V_0 has valency 3, distinct vertices in $V_1 \setminus \{1\}$ are adjacent to distinct vertices in V_0. Suppose that there are three of them, say 3, 5, 7, respectively adjacent to 2, 4, 6. We have then four edges, namely 1 2, 3 2, 5 4 and 7 6 in three connected components, a contradiction. Thus $|V_1| = 3$. □

Corollary 4.2. *Let t_1 and t_2 be the number of minimal vectors of types 1 and 2 respectively. Then $t_1 + t_2 \leq 9$, where equality holds only if the graph of the relation $\equiv$ is the complete bipartite graph ($(t_1, t_2) = (0, 9)$) or if it consists of two non-adjacent nodes of valency 3 and their common adjacent vertices ($(t_1, t_2) = (3, 6)$).*

Proof. Since by Proposition 3.1 we know that $t_1 \leq 4$, we only need to consider the graphs (in the sense of the theorem) with $t_2 \geq 5$ edges. We first note that if i is an isolated point of the graph, $e - e_i$ cannot be minimal: we could extract from the $t_2 \geq 5$ edges of the graph three disjoint ones, or two secant and a third one disjoint, which, together with i, would contradict Proposition 3.3. We now consider the case of a pentagon say $(1, 2, 3, 4, 5, 1)$. If there are 4 vectors of type 1, three of them correspond to consecutive vertices, say $e - e_1$, $e - e_2$ and $e - e_3$, which together with $e - e_4 - e_5$, contradicts again 3.3. Thus $t_1 \leq 3$ and $t_1 + t_2 \leq 8$. The other graphs to consider are included in the complete bipartite graph associated with, say, the partition $\{2, 4, 6\} \cup \{1, 3, 5\}$. We first consider a path of length 5, say $1-2-3-4-5-6$, and suppose

$e - e_i$ minimal ($i = 1, \dots, 6$). Then $i = 1$ is not possible, because the four sets $I_1 = \{1\}$, $I_2 = \{2, 3\}$, $I_3 = \{4, 5\}$, $I_4 = \{5, 6\}$ contradict Proposition 3.3. The same argument, with $I_2 = \{1, 2\}$ instead of $\{2, 3\}$, forbids $i = 3$. So the only possible values of i are $i = 2$ and $i = 5$, and $t_1 \leq 2$, $t_1 + t_2 \leq 7$. Now consider the cycle $(1, 2, 3, 4, 5, 6, 1)$. By considering the path $1-2-3-4-5-6$, we see that $e - e_1$ is not minimal, and since all vertices play the same role, we conclude that $t_1 = 0$. This conclusion extends to any subgraph of the complete bipartite graph containing such a cycle, i.e. the complete graph itself, and the ones obtained by suppressing one edge, two disjoint edges or three pairwise disjoint edges.

One more graph with 5 edges contains no node of valency 3: the disjoint union of a cycle of length 4, say $(1, 2, 3, 4)$, and a path of length 1. Suppose $e - e_1$ minimal; the four sets $I_1 = \{1\}$, $I_2 = \{2, 3\}$, $I_3 = \{3, 4\}$, $I_4 = \{5, 6\}$ contradict 3.3. Thus there are at most two minimal vectors of type 1, namely $e - e_5$ and $e - e_6$, and $t_1 + t_2 \leq 7$.

We are left with graphs which contain at least one node of valency 3, say 1, with adjacent vertices $2, 4, 6$. If $e - e_i$ is minimal, we must have $i \in \{1, 2, 4, 6\}$ (otherwise, the four sets of indices $\{1, 2\}$, $\{1, 4\}$, $\{1, 6\}$ and $\{i\}$ should contains indices of weight one–respectively 2, 4, 6 and i– but also an index of weight 3, which contradicts Proposition 3.3). Note that the four values $i = 1, 2, 4, 6$ are never simultaneously possible: since there are more than four edges, one of the vertices $2, 4, 6$, say 2, has another adjacent vertix, say 3. Then if $e - e_1$, $e - e_4$, $e - e_6$ were minimal vectors, the sets $\{1\}$, $\{4\}$, $\{6\}$ and $\{2, 3\}$ would contradict Proposition 3.3. We then have $t_1 \leq 3$, which completes the case $t_2 = 5$.

If there are, in the graph we consider, two adjacent nodes of valency 3, say 1 and 2, the only possible minimal vectors of type 1 are thus $e - e_1$ and $e - e_2$. In particular, the graph of $t_2 = 7$ edges obtained by suppression from the complete bipartite graph two secant edges, say $3 - 2$ and $3 - 4$ contains three nodes of valency 3, namely 1, 5 and 6, where 6 is adjacent to 1 and 5. The unique minimal vector of type 1 is thus $e - e_6$, and $t_1 + t_2 \leq 8$. This completes the case $t_2 = 7$.

We are left with 3 non-isomorphic graphs with 6 edges. If it is obtained by suppressing (from the complete graph) the three edges of a path of length 3, it contains two adjacent nodes of valency 3, and $t_1 \leq 2$, as announced. The same conclusion is valid for the graph obtained by suppression of three edges, two of them secant, for instance $4 - 5$, $5 - 6$ and $2 - 3$. The resulting graph contains the disjoint union of the cycle (14361) with the edge $2 - 5$, and we have seen that the only possible minimal vectors are $e - e_2$ and $e - e_5$, and again $t_1 \leq 2$. But for the graph obtained by suppressing three secant edges, say $5 - 2$,

$5-4$ and $5-6$, it contains two non-adjacent nodes, and it is consistent with $t_1 = 3$ minimal vectors of type 1, namely $e - e_2$, $e - e_4$, $e - e_6$. The proof of the corollary is now complete. □

5. Configurations of three vectors of type $p \geq 3$

The graph we consider is that of the relation $\sim$ introduced in Proposition 3.5: $I_k \sim I_{k'}$ if $I_k \cap I_{k'}$ contains an index of weight 2.

Proposition 5.1. *Let $x_1 = e - \sum_{i \in I_1} e_i$, $x_2 = e - \sum_{i \in I_2} e_i$, $x_3 = e - \sum_{i \in I_3} e_i$ be three vectors of the same type $p \geq 3$. We suppose that $W_3 = \cap I_k$ is not empty. Then, if $(p, n) \neq (4, 8)$, one at least of the sets I_1, I_2 and I_3 has no index of weight one, and the $\sim$-graph is a path.*

Proof. For all $k = 1, 2, 3$ we put $a_k = |W_1 \cap I_k|$, $b_k = W_2 \cap_{h \neq k} I_h$, and $c = |W_3|$. We have $p = |I_k| = a_k + (|W_2| - b_k) + c$, and thus the $a_k - b_k$ have a common value $p - |W_2| - c$.

• Suppose first $a_k \geq 1$ for all k. Then by Proposition 3.3, we have $a_1 = a_2 = a_3 = c = 1$. Thus, the b_k have a common value $|W_2|/3$, where $2|W_2| = 3p - |W_1| - 3|W_3| = 3p - 6$; p is even, $|W_2| = \frac{3(p-2)}{2}$, and the $b_k = \frac{p-2}{2}$ are non-zero. We conclude that the $\sim$-graph is a cycle, and by Proposition 3.5 we must have $n \leq m + 1$, where $m = \sum |W_i| = \frac{3}{2}p + 1$. The unique solution for the inequalities $2p \leq n \leq \frac{3}{2}p + 2$ is $p = 4$, $n = 8$.

• Now we suppose $a_1 = 0$, and thus $p = b_2 + b_3 + c$. Since $|I_1 \cap I_2| = b_3 + c$ is $< p$, b_2 is non-zero, and so is b_3. From $|W_2| = b_1 + p - c$ follow $3p - m = |W_2| + 2|W_3| = b_1 + p + c$ and $m = 2p - c - b_1$. If the graph were a cycle, i.e. if $b_1 \geq 1$, we should obtain $m \leq 2p - 2$ (since $c = |W_3| \geq 1$), and thus $m + 1 < 2p$, a contradiction with Proposition 3.5. We conclude that the graph is the path $I_2 \sim I_1 \sim I_3$. □

Corollary 5.2. *We suppose $(p, n) \neq (4, 8)$, and we consider three distinct minimal vectors $x_0 = x_{I_0}, x = x_I$ and $x' = x_{I'}$ of the same type $p \geq 3$ such that I and I' both intersect I_0. We put*

$$I = A \cup X \text{ with } \quad A = I \cap I_0, \quad X = I \smallsetminus A\,,$$

$$I' = A' \cup X' \text{ with } \quad A' = I' \cap I_0, \quad X' = I \smallsetminus A\,.$$

Then:

(i) The $\sim$-graph (I, I', I_0) is a cycle if and only if $A \cap A' = \emptyset$ and $X \cap X' \neq \emptyset$.

(ii) If A and A' satisfy an inclusion, so do X and X'.

(iii) If A and A' satisfy no inclusion and if (I, I', I_0) is not a cycle, then X and X' are disjoint; if moreover $A \cap A' \neq \emptyset$, then $I_0 = A \cup A'$.

Proof. The sets of indices of weight one in I_0, I and I' are respectively $I_0 \setminus (A \cup A')$, $X \setminus X'$ and $X' \setminus X$; the sets of indices of weight two in $I_0 \cap I$, $I_0 \cap I'$ and $I \cap I'$ are respectively $A \setminus A'$, $A' \setminus A$ and $X \cap X'$; eventually the set W_3 of indices of weight three is $A \cap A'$.

(i) We see directly that (I, I', I_0) is a cycle if and only if A and A' satisfy no inclusion and X and X' intersect. From Hypothesis $(p, n) \neq (4, 8)$, Proposition 5.1 shows that it can only happen when $A \cap A' = \emptyset$.

(ii) Assume $A' \subset A$. Then $W_3 = A'$ is not empty. From Proposition 5.1, one at least of the sets I, I' and I_0 contains no index of weight one. Since $I_0 \setminus (A \cup A') = I_0 \setminus A$ is not empty (since $I \neq I_0$ have the same cardinality), X and X' satisfy an inclusion, namely $X \subset X'$ (since $|A| + |X| = |A'| + |X'| = p$). In particular, if $A = A'$, we have $X \subset X'$ and $|X| = |X'|$, thus $X = X'$ and $I = I'$.

(iii) Now we assume that A and A' satisfy no inclusion, i.e. $I \sim I_0$ and $I' \sim I_0$. As already noted in (i), we must have $I \nsim I'$, i.e. $X \cap X' = \emptyset$. If moreover $A \cap A' \neq \emptyset$, by Proposition 5.1 one at least of the sets $W_1 \cap I = X$, $W_1 \cap I' = X'$ and $W_1 \cap I_0 = I_0 \setminus (A \cup A')$ is empty, thus $I_0 = A \cup A'$. □

The end of the section is devoted to the special case $(p, n) = (4, 8)$.

Proposition 5.3. *If $n = 8$, then $t_4 \leq 6$.*

Proof. Indeed, since $n = 2p$ we may and shall assume that all index sets contain 1. Let $I_0 = \{1, 2, 3, 4\}$ be one of them; from 3.4 we know that there is at most one $I \neq I_0$ for a given $I \cap I_0$. Let I be such that $|I \cap I_0| = 3$. For instance, put $I_1 = \{1, 2, 3, 5\}$, and let $I_2 = \{1, 2, 4, a\}$, with $a \in \{5, 6, 7, 8\}$. Actually, if $a = 5$, the configuration $\{I_0, I_1, I_2\}$ is a cycle in the sense of 3.5, which is absurd since $n = 8 > 5 + 1$. Thus for instance $I_2 = \{1, 2, 4, 6\}$ and similarly $I_3 = \{1, 3, 4, 7\}$. Now, let x_I be a minimal vector such that $|I \cap I_0| = 1$, i.e. $I = \{1, a, b, c\}$, with $\{a, b, c\} \subset \{5, 6, 7, 8\}$. Actually, by Proposition 3.3 we must have $\{a, b, c\} = \{5, 6, 7\}$. Otherwise, if for instance $5 \notin \{a, b, c\}$, the configuration $\{I_0, I_1, I\}$ would contain too many indices of weight 1. Now we have $I \sim I_k$ for $k = 1, 2, 3$ and the configuration $\{I_1, I_2, I_3, I\}$ contradicts Proposition 3.5 ($n = 8, m = 7, |W_2| = 6 > 3$). We conclude that there are at most 3 minimal vectors x_I such that $|I \cap I_0| = 1$ or 3. Now, we shall prove that there are at most 2 minimal vectors x_I such that $|I \cap I_0| = 2$. Otherwise, let $I = \{1, 2\} \cup X$, $I' = \{1, 3\} \cup X'$ and $I" = \{1, 4\} \cup X"$ be three solutions (X, X' and $X"$ subsets of $\{5, 6, 7, 8\}$). In their configuration $(I, I', I")$, 2, 3 and 4 have weight 1, and by 5.1 the elements of X, X' and $X"$ must have weight $\neq 1, 3$, i.e. they have weight 2, wich leads (up to permutation) to $I = \{1, 2, 5, 6\}$,

$I' = \{1,3,6,7\}$ and $I" = \{1,4,5,7\}$. Now the configuration $I_0, I, I', I"$ satisfies the hypotheses of Proposition 3.5, with $m = 7$ and $n = 8$, but $|W_2| = 6$, a contradiction. $\square$

Taking into account Proposition 5.3, we discard in the next sections the case $(p,n) = (4,8)$.

6. Families without cycles of length 3

Theorem 6.1. *Let $\{x_I, I \in \mathcal{F}\}$ be a set of minimal vectors of the same type $p \geq 3$, such that $\mathcal{F}$ contains no cycles of length 3.*

If $p \geq 4$ and $n \geq 2p+2$, then $|\mathcal{F}| \leq p+6$.

If $p \geq 4$ and $n = 2p+1$, or $p = 3$ and $n \geq 8$, then $|\mathcal{F}| \leq p+5$.

If $(p,n) = (3,7)$, then $|\mathcal{F}| \leq 7$.

If $p \geq 3$, $p \neq 4$ and $n = 2p$, then $|\mathcal{F}| \leq p+1$.

[Note that the bound $p+6$ is indeed reached, as checked for $(p,n) = (4,10), (5,12), \ldots$.]

The whole section is devoted to the proof of this theorem. We first consider the case when all elements of $\mathcal{F}$ intersect a given one, which includes the case $p = \frac{n}{2}$, since then we may prescribe that all I contain a given index.

Proposition 6.2. *We suppose that there exists $I_0 \in \mathcal{F}$ such that for all $I \in \mathcal{F}$, $I \cap I_0 \neq \emptyset$.*

If $p \geq 4$ (resp. $p = 3$) and $n \geq 2p+1$, then $|\mathcal{F}| \leq p+3$ (resp. 5); if $p \geq 3$ and $n = 2p$, then $|\mathcal{F}| \leq p+1$.

Proof. For $I \in \mathcal{F}$, we write

$$I = A \cup X, \text{ with } A = I \cap I_0 \neq \emptyset \text{ and } X = I \smallsetminus A\,.$$

From 3.4 it follows that I is uniquely specified by A (or equivalently by X). We shall now describe the set

$$\mathcal{F}_0 = \{I \cap I_0,\, I \in \mathcal{F}\}$$

in one-to-one correspondence with $\mathcal{F}$, and prove that it consists of one or two totally ordered sequences, except in the following case.

Lemma 6.3. *Let $I = A \cup X$, $I' = A' \cup X'$ and $I" = A" \cup X"$ be three elements of $\mathcal{F} \smallsetminus \{I_0\}$ such that A, A' and $A"$ satisfy no pairwise inclusions. Then $|\mathcal{F}| = 4$.*

Proof of the lemma. From Corollary 5.2 we see that the sets X, X' and X'' are pairwise disjoint. First, we prove that A, A' and A'' are pairwise disjoint. Otherwise suppose for instance $A \cap A' \neq \emptyset$, and thus, by Corollary 5.2, $I_0 = A \cup A'$, i.e. $I_0 \smallsetminus A \subset A'$; since $A"$ is not included in A', it is no more included in $I_0 \smallsetminus A$ i.e. $A" \cap A \neq \emptyset$; and of course we also have $A" \cap A' \neq \emptyset$. In the configuration $\{I, I', I"\}$, the indices in X, X' and X' have weight one, thus by Proposition 5.1 $I \cap I' \cap I" = \emptyset$, so that the indices in $A \cap A'$, $A' \cap A"$ and $A \cap A"$ have weight 2, and $\{I, I', I"\}$ is a cycle, a contradiction: A, A' and $A"$ are pairwise disjoint as announced. Now in the configuration $\{I, I', I", I_0\}$, the sets of weight one in I, I', $I"$ and I_0 are respectively equal to $X \neq \emptyset$, $X' \neq \emptyset$, $X" \neq \emptyset$ and $I_0 \smallsetminus (A \cup A' \cup A")$. If this last set were not empty, we should obtain from Proposition 3.3 $|X| = |X'| = |X"| = |I_0 \smallsetminus (A \cup A' \cup A")| = 1$, thus $|A \cup A' \cup A"| = 3(p-1) = p-1$, i.e. $p = 1$, absurd. So $I_0 = A \cup A' \cup A"$ is a partition of I_0. Let $J = B \cup Y$ be another element of $\mathcal{F}$. Then $B = J \cap I_0$ satisfies an inclusion with at most one of the sets A, A', $A"$ (since these sets are mutually disjoint, and so are their complements X, X' and $X"$). Thus B satisfies no inclusion with at least two among A, A', $A"$, say A and A'. Thus $I_0 = A \cup A' \cup B$ is a partition of I_0, and $B = A"$, a contradiction. Thus $\mathcal{F} = \{I_0, I, I', I"\}$. $\square$

If $\mathcal{F}_0 = \{I \cap I_0,\ i \in \mathcal{F}\}$ is a totally ordered family, we have $|\mathcal{F}_0| = |\mathcal{F}| \leq |I_0| = p$, and Proposition 6.2 is proved. The same conclusion holds in the situation of Lemma 6.3. We therefore consider in $\mathcal{F}$

$$I = A \cup X \ \text{ and } \ J = B \cup Y \quad \text{such that } \ A \not\subset B \quad \text{and } \ A \not\supset B\,,$$

and may suppose that for any $K = C \cup Z \in \mathcal{F}$, $C \in \mathcal{F}_0$ satisfies an inclusion with A or B, or equivalently (by 5.2), Z satisfies an inclusion with X or Y, which are disjoint. If Z satisfies an inclusion with both X and Y, i.e. if $Z \supset X \cup Y$, then by Proposition 5.1 applied to $\{I, J, K\}$ (since the indices of $C \neq \emptyset$ have weight 3, and those of $A \smallsetminus B \neq \emptyset$ and $B \smallsetminus A \neq \emptyset$ have weight 1), Z coincides with $X \cup Y$, and C with $A \cap B$.

Now choose a pair (A, B) with $|A|$ and $|B|$ minimal: if $C \neq A \cap B$ satisfies an inclusion with A (resp. B), it contains A (resp. B). [If $C \subsetneq A$, from the minimality of the pair (A, B), C satisfies an inclusion with B too, and we just saw that $C = A \cap B$.] Thus, apart from I_0 and (possibly) $A \cap B$, $\mathcal{F}_0$ is union of two disjoint sets

$$\mathcal{A} = \{C \in \mathcal{F}_0 \mid A \subset C \subsetneq I_0\} \quad \text{and} \quad \mathcal{B} = \{C \in \mathcal{F}_0 \mid B \subset C \subsetneq I_0\}\,.$$

These sets are totally ordered by inclusion, as we now prove. Consider for instance in $\mathcal{F}$ two distinct elements $K = C \cup Z \in \mathcal{F}$ and $K' = C' \cup Z' \in \mathcal{F}$ with $C \supset A$ and $C' \supset A$, i.e. by 5.2, Z and Z' included in X. Thus Z and Z' are disjoint from Y, and by 5.2 again, B satisfies no inclusion with C or C'. Since $|\mathcal{F}| \geq 5$, Lemma 6.3 implies that C and C' satisfy an inclusion: the set $\mathcal{A}$ is totally ordered by inclusion, and so is $\mathcal{B}$. We then have

$$|\mathcal{A}| \le p - |A| \quad \text{and} \quad |\mathcal{B}| \le p - |B|\,.$$

We have to consider two cases:

case $A \cap B \neq \emptyset$, *in particular* $n = 2p$. It follows from Corollary 5.2 that $I_0 = A \cup B$ and then $|A| + |B| = p + |A \cap B| \ge p + 1$. It implies $|\mathcal{A}| + |\mathcal{B}| \le 2p - |A| - absB \le p - 1$, and taking into account I_0 and $A \cap B$, $\mathcal{F} = \mathcal{F}_0 \le p + 1$ as required.

case $A \cap B = \emptyset$. We then have $\mathcal{F}_0 = \{I_0\} \cup \mathcal{A} \cup \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are totally ordered sequences $A = A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_k \subsetneq I_0$ and $B = B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_h \subsetneq I_0$ (every pair (A_i, B_j) without inclusion),with for instance $1 \le k \le h \le p - 1$. We then have $|\mathcal{F}| \le 1 + h + k$. If $k = 1$, we obtain $|\mathcal{F}| \le p + 1$, and Proposition 6.2 is proved in this case. The same conclusion holds if $h \le 2$, for instance if $p = 3$, since then we obtain $|\mathcal{F}| \le 5$. We thus suppose $h \ge 3$ and $k \ge 2$. and consider the four elements of $\mathcal{F}$ corresponding to A, B, A_3 and B_2, say $I = A \cup X$, $J = B \cup Y$, $I_3 = A_3 \cup X_3$ and $J_2 = B_2 \cup Y_2$, with X and Y disjoint and $X_3 \subsetneq X$ and $Y_2 \subsetneq Y$. Their respective subsets of weight 1 are $X \smallsetminus X_3 \neq \emptyset$, $Y \smallsetminus Y_2 \neq \emptyset$, $A_3 \smallsetminus (A \cup B_2)$ and $B_2 \smallsetminus (A_3 \cup B)$. If $A_3 \cap B_2$ were empty, we should have $|A_3 \smallsetminus (A \cup B_2)| = |A_3 \smallsetminus A| \ge 2$ and $|B_2 \smallsetminus (A_3 \cup B| = |B_2 \smallsetminus B| \ge 1$, a contradiction with Proposition 3.3. Thus $A_3 \cap B_2$ is not empty, and from 5.2, it follows that $|A_3| + |B_2| \ge p + 1$. Now from $h = |\mathcal{A}| \le 2 + (p - A_3)$ and $k \le 1 + (p - B_2)$ we obtain $\mathcal{F} \le 4 + 2p - (|A_3| + |B_2|) \le p + 3$ as required. □

We know that (for $p > 1$), the family $\mathcal{F}$ contains at most three pairwise disjoint elements. We examine now this case.

Proposition 6.4. *If* $\mathcal{F}$ *contains three pairwise disjoint elements, then* $\mathcal{F} \le p + 3$.

Proof. This will follow from the more precise result 6.5, for which we need some more notation.

Let I_1, I_2, I_3 be three elements of $\mathcal{F}$ pairwise disjoint. For every $I \in \mathcal{F}$ distinct from the I_j we consider the partition $I = A_1 \cup A_2 \cup A_3 \cup X$, where $A_j = I \cap I_j$. Actually, X is empty. Otherwise, we could apply Proposition 3.5 to the subset $\{I_1, I_2, I_3, I\} \in \mathcal{F}^4$, whose sets of indices of weight one $I_j \smallsetminus A_j, j = 1, 2, 3$ and X should have just one element, and I should have $p = 3(p-1)+1$ elements, i.e. $p = 1$, a contradiction. We thus have

$$I = A_1 \cup A_2 \cup A_3, \qquad \text{where } A_j = I \cap I_j.$$

We introduce the following subsets of $\mathcal{F}$:

For $i = 1, 2, 3$, $\mathcal{F}_i$ is the set of $I \in \mathcal{F}$ with only A_i non-empty. We have just proved that $\mathcal{F}_i = \{I_i\}$.

For $1 \le i < j \le 3$, $\mathcal{F}_{ij}$ is the set of $I \in \mathcal{F}$ with only A_i and A_j non-empty.

Eventually, $\mathcal{F}_{123}$ is the set of $I \in \mathcal{F}$ intersecting I_1, I_2 and I_3.

Lemma 6.5. *The four subsets $\mathcal{F}_{ij}$ and $\mathcal{F}_{123}$ are empty but one. We have $|\mathcal{F}_{ij}| \le p - 1$ and $|\mathcal{F}_{123}| \le 3$.*

Proof of 6.5. We may suppose that $|\mathcal{F}| \ge 5$. Let $I \ne I'$ be two elements of $\mathcal{F}$ distinct from I_1, I_2, I_3. There exists $i \in \{1, 2, 3\}$ such that both A_i and A'_i are non-empty, for instance we suppose A_1 and A'_1 non-empty, and we are in the situation described by Corollary 5.2 with I_1 in the rôle of I_0. We have to consider two cases.

Case 1. $A'_1 \subsetneq A_1$. Then by 5.2 we have $A_2 \cup A_3 \subset A'_2 \cup A'_3$, i.e. $A_2 \subset A'_2$ and $A_3 \subset A'_3$. As I is distinct from I_1, A_2 for instance is non-empty, and so is A'_2. Then by 5.2, the inclusion $A_2 \subset A'_2$ implies now $A'_3 \subset A_3$, and thus $A_3 = A'_3$. Since I and I' are distinct, from 3.4 we conclude that A_3 and A'_3 are empty, i.e. that I and I' both lie in $\mathcal{F}_{12}$.

Case 2. A_1 and A'_1 satisfy no inclusion. Then by 5.2 $A_2 \cup A_3$ and $A'_2 \cup A'_3$ are disjoint, i.e $A_2 \cap A'_2 = A_3 \cap A'_3 = \emptyset$. Then I and I' do not belong to distinct $\mathcal{F}_{ij}$, as we now prove. If (I, I') lies in $\mathcal{F}_{12} \times \mathcal{F}_{13}$, Proposition 3.3, applied to the four elements I, I', I_2, I_3, gives $|I_2 \smallsetminus A_2| = |I_3 \smallsetminus A'_3| = 1$, thus A_1 and A'_1 are disjoint singletons, and the same proposition applied to $\{I, I', I_1, I_3\}$ gives, since $p \ge 3$ ($I_1 \supsetneq A_1 \cup A'_1$) $|A_2| = 1$ and thus $|I| = p = 1 + 1$, a contradiction.

We may assume for instance that A_2 and A'_2 are both non-empty. Then permuting I_1 and I_2 we conclude that A_1 and A'_1 are disjoint two. We then have, for $i = 1, 2, 3$, $|A_i| + |A'_i| \le p$. Since $\sum_i (|A_i| + |A'_i|) = |I| + I'| = 2p$, we have two possibilities.

1) $|A_1| + |A'_1| = |A_2| + |A'_2| = p$, and thus $A_3 = A'_3 = \emptyset$: I and I' lie in $\mathcal{F}_{12}$. Note that in this case, $\mathcal{F}_{12}$ reduces to the pair I, I', since $I' = (I_1 \smallsetminus A_1) \cup (I_2 \smallsetminus A_2)$ is uniquely determined by I.

2) Otherwise, there are at least two sums $|A_i| + |A'_i| < p$, say for $i = 2$ and $i = 3$. By use of Proposition 3.3 applied to the set $\{I, I', I_2, I_3\}$, we obtain (since $A_1 \cap A'_1 = \emptyset$) $|A_1| = |A'_1| = |I_2 \smallsetminus (A_2 \cup A'_2)| = |I_3 \smallsetminus (A_3 \cup A'_3)| = 1$; since the third sum $|A_1| + |A'_1| = 2$ is also $< p$, we obtain $|A_i| + |A'_i| = p - 1$ for all i. We then have $p = 3$, and for all i, A_i and A'_i are disjoint singletons. Thus I and I' both lie in $\mathcal{F}_{123}$. [Note that Proposition 3.5 applied to the star I, I_1, I_2, I_3 shows that in this case, $n = 9$ or 10]

We conclude that two distinct elements I and I' of $\mathcal{F}$ belong to the same subset $\mathcal{F}_{ij}$ or $\mathcal{F}_{123}$, therefore only one of them is non-empty. Moreover, the elements of $\mathcal{F}_{123}$ are pairwise disjoint, thus by 3.4 there are at most three of them. Eventually, if $\mathcal{F}_{ij}$ is not empty, it consists either of a disjoint pair $(I, (I_i \cup I_j) \smallsetminus I)$, or of at most $p-1$ elements $I = A_i \cup A_j$, where the set $\{A_i\}$ is totally ordered by inclusion. This completes the proof of 6.5 and thus of Proposition 6.4.

We now come back to the proof of Theorem 6.1. Taking into account the result of 6.2 and 6.4, we may and will assume now that $\mathcal{F}$ contains two disjoint elements, say I_1 and I_2, such that every $I \in \mathcal{F}$ intersects at least one of them.

In other terms, there is a partition $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_{1,2}$, with

$\mathcal{F}_1 = \{I \in \mathcal{F} \quad | \quad I \cap I_1 \neq \emptyset \text{ and } I \cap I_2 = \emptyset\}$,

$\mathcal{F}_2 = \{I \in \mathcal{F} \quad | \quad I \cap I_2 \neq \emptyset \text{ and } I \cap I_1 = \emptyset\}$,

$\mathcal{F}_{1,2} = \{I \in \mathcal{F} \quad | \quad I \cap I_1 \neq \emptyset \text{ and } I \cap I_2 \neq \emptyset\}$.

Since by Proposition 6.2 we know that $|\mathcal{F}_1 \cup \mathcal{F}_{1,2}| \leq p+3$ ($p+2$ if $p=3$), the proof of Theorem 6.1 will result from the following proposition.

Proposition 6.6. *We have* $\min(|\mathcal{F}_1|, |\mathcal{F}_2)|) \leq 3$, *where equality holds only when* $n \geq 2p+2$.

We keep the notation $I = A_1 \cup X$ for an element $I \neq I_1$ in $\mathcal{F}_1$, where $A_1 = I \cap I_1$ and $X = I_1 \smallsetminus A_1$, and similarly $J = B_2 \cup Y$ ($B_2 = J \cap I_2$, $Y = I_2 \smallsetminus B_2$) for an element $J \neq I_2$ of $\mathcal{F}_2$.

Lemma 6.7. *Let* $I = A_1 \cup X$ *be an element of* $\mathcal{F}_1 \smallsetminus \{I_1\}$. *Then there is* at most one $J = B_2 \cup Y \in \mathcal{F}_2$ *such that* Y *satisfies no inclusion with* X, *and this may occur only when* $|X| = 1$ *(and obviously* Y *also is a singleton).*

Proof of the lemma. Let $J = B_2 \cup Y \in \mathcal{F}_2$ be distinct from I_2 such that $Y \not\subset X$ and $Y \not\supset X$. With respect to the set $\{I_1, I_2, I, J\}$, the subsets of indices of weight one in I_1, I_2, I, J are respectively are $I_1 \smallsetminus A_1$, $I_2 \smallsetminus B_2$, $X \smallsetminus Y$ and $Y \smallsetminus X$, all of them non-empty, and by Proposition 3.3 all of them singletons. From $|A_1| = |B_2| = p-1$, follows $|X| = |Y| = 1$. Now, let $J' = B'_2 \cup Y'$ be another solution in $\mathcal{F}_2$, i.e. with Y' singleton distinct from X, and also from Y since $J' \neq J$. The subsets of indices of weight one in I_1, I, J, J' respectively are $I_1 \smallsetminus A_1 \neq \emptyset$, X, $(B_2 \smallsetminus B'_2) \cup Y$, $(B'_2 \smallsetminus B_2) \cup Y'$, the last two subsets with $p-1 > 1$ elements, which contradicts Proposition 3.3. The solution J is unique. $\square$

Proof of Proposition 6.6. For it we may and will assume that $\mathcal{F}_1$ contains at least two elements $I = A_1 \cup X$ and $I' = A_1' \cup X'$ distinct from I_1. We fix such a pair I, I' and suppose for instance $|X| \le |X'|$. We now prove that there is in $\mathcal{F}_2$ at most one $J = B_2 \cup Y$ with Y satisfying an inclusion with X.

1) First suppose that A_1 and A_1' satisfy an inclusion, for instance $A_1' \subsetneqq A_1$. Then with respect to the set $\{I, I', J\}$, I and J have indices of weight one (those of $A_1 \smallsetminus A_1'$ and B_2 at least). Since $I \cap I' \cap J = X \cap Y$ is not empty, Proposition 5.1 implies that I' has no index of weight one, i.e. $X' \subset X \cup Y$. The inclusion between X and Y is thus $X \subset Y$, and we conclude that Y contains X and X'. Now, let $J' = B_2' \cup Y'$ be another solution with Y' satisfying an inclusion with X. Then Y and Y' containing X' should intersect, thus satisfy an inclusion, and B_2' and B_2 too. Hence we might exchange the rôle of the pairs (I, I') and (J, J'), and conclude that X and X' must contain Y and Y', thus $X = X' = Y' = Y$, a contradiction.

2) We now suppose that A_1 and A_1' satisfy no inclusion. We may have two types of solutions $J = B_2 \cup Y$, with Y satisfying an inclusion with X.

- *Type I: Y satisfies an inclusion with X' too.* Then we must have $Y \supset X \cup X'$ since X and X' are disjoint. Actually, if Y contains strictly $X \cup X'$, we may apply Proposition 3.3 to the set $\{I, I', J, I_2\}$, with sets of indices of weight one $A_1 \smallsetminus A_1'$, $A_1' \smallsetminus A_1$, $I_2 \smallsetminus B_2$ and $Y \smallsetminus X \cup X'$, and conclude $|B_2| = p - 1$, thus $|Y| = 1$, a contradiction. Hence we have $Y = X \cup X'$, which determines entirely J in $\mathcal{F}_2$.
- *Type II: Y satisfies no inclusion with X'* . We know by Lemma 6.7 that such a solution is unique and implies $|Y| = |X'| = 1$, and thus $|X| = 1$ too ($|X| \le |X'|$). More precisely since Y and X satisfy an inclusion, we have $Y = X = \{x\}$ and $X' = \{x'\}$, $x' \neq x$.

It remains to prove that we cannot have simultaneously in $\mathcal{F}_2$ solutions of types I and II. We then suppose $X = \{x\}$ and $X' = \{x'\}$, $x' \neq x$, and we consider in $\mathcal{F}_2$ an element $J = B_2 \cup Y$ of the first type, i.e. with $Y = \{x, x'\}$, and an element of the second type $J' = B_2' \cup Y'$ with $Y' = \{x\}$. Since $Y' \subset Y$, we have $B_2 \subset B_2'$. We may thus apply the part 1) to I, J and J' (since X satisfies an inclusion with Y), and conclude that X must contain Y and Y', a contradiction.

We have then proved that in every case there is at most one element $J = B_2 \cup Y$ in $\mathcal{F}_2$ such that Y satisfies an inclusion with X, and by Lemma 6.7 we obtain $|\mathcal{F}_2| \le 1 + 1 + 1$.

In order to complete the proof of 6.6, it remains to observe that if $\mathcal{F}_2$ contains (apart from I_2) two elements $J = B_2 \cup Y$ and $J' = B_2' \cup Y'$, then $|Y \cup Y'| \geq 2$ (since $Y \neq Y'$ by 6.7 and 5.2) and therefore $|\cup_{I\in\mathcal{F}} I| \geq |I_1| + |I_2| + 2 = 2p + 2$.

□

7. Families with cycles of length 3

Theorem 7.1. *Let $\{x_I, I \in \mathcal{F}\}$ be a set of minimal vectors of the same type $|I| = p \geq 3$. We suppose that $\mathcal{F}$ contains a cycle of length 3, and that $(p, n) \neq (4, 8)$. Then*

(1) the dimension n of the lattice satisfies $2p + 1 \leq n \leq 3p - 2$;

(2) we have $|\mathcal{F}| \leq n$, and even, if $n = 3p - 2$ and $p \geq 4$, $|\mathcal{F}| \leq p + 2$.

The section is devoted to the proof of the theorem. Let (I_1, I_2, I_3) be in $\mathcal{F}$ a fixed cycle for the relation $\sim$. We use for this cycle the notation and rules of Section 2. In particular $W_k \subset \cup_{h=1,2,3} I_h$ is the set of indices of weight k, $k = 1, 2, 3$, and $m = |I_1 \cup I_2 \cup I_3|$.

(1) Since $(p, n) \neq (4, 8)$ we know by 5.1 that $W_3 = I_1 \cap I_2 \cap I_3$ is empty, which allows us to prescribe $n \geq 2p + 1$. We thus have

$$W_2 = E_{12} \cup E_{23} \cup E_{13} \quad | \quad E_{ij} = I_i \cap I_j \neq \emptyset$$

since $I_i \sim I_j$. From the relations $m = \sum_k |W_k|$ and $3p = \sum_k k|W_k|$ we obtain

$$|W_2| = 3p - m\,;$$

hence $|W_2| \geq 3$ reads $m \leq 3p - 3$, and the inequality $n \leq 3p - 2$ follows from 3.5 (n is equal to m or $m + 1$).

For every permutation (i, j, k) of $\{1, 2, 3\}$, let

$$E_k = I_k \smallsetminus (E_{ki} \cup E_{kj})$$

denote the set of indices of weight 1 in I_k. The condition $|I_k| = p$ reads $p = |E_k| + |E_{ki}| + |E_{kj}|$, which implies $|E_k| \leq p - 2$, and even $|E_k| = p - 2$ if $n = m + 1$. Moreover it proves that $|E_k| - |E_{ij}|$ does not depend on k:

$$\Delta = |E_k| - |E_{ij}| = p - |W_2| = m - 2p \geq 0\,,$$

where equality holds when $n = m + 1$ and thus (since $|W_2| = 3$ by 3.5) $(p, n) = (3, 7)$. We thus have $|E_k| \geq 1$, with equality if and only if $(p, n) = (3, 7)$.

In the following,

$$I = A_1 \cup A_2 \cup A_3 \cup A_{12} \cup A_{23} \cup A_{13} \cup (I \cap \{m + 1\})\,,$$

$$\text{with} \ \mid A_i = I \cap E_i,\ A_{ij} = I \cap E_{ij}$$

denotes an element of $\mathcal{F}$ distinct from I_1, I_2, I_3.

The discussion below is based on the A_{ij}, starting with the case when they are empty.

Lemma 7.2. *If I intersects no E_{ij}, then $n = m + 1$ belongs to I and (p, n) is equal to $(3, 7)$ or $(4, 10)$. In the first case there is at most one such I, say $I = E_1 \cup E_2 \cup \{7\}$; in the second case there are at most two of them, say $I = \{a_1, a_2, a_3, 10\}$ and $I' = \{a'_1, a'_2, a'_3, 10\}$, where $E_i = \{a_i, a'_i\}$.*

Proof. By assumption I is of the form $I = A_1 \cup A_2 \cup A_3 \cup (I \cap \{m+1\})$ and the condition $|I| = p$ reads $p = |A_1| + |A_2| + |A_3| + \varepsilon$, where $\varepsilon = |I \cap \{m + 1\}|$. From $|A_i| \le |E_i| \le p - 2$ it follows that at most two A_i are non-empty, say A_1 and A_2. Then, (I, I_1, I_2) *is* a cycle. Put $m_{12} = |I \cup I_1 \cup I_2|$. By Proposition 3.5, we must have $n = m_{12}$ or $m_{12} + 1$, and thus $|m - m_{12}| \le 1$, where $m_{12} - m = \varepsilon - |E_3 \smallsetminus A_3|$.

Case $m_{12} = m + 1$, i.e. $n = m + 1$, $\varepsilon = 1$ and $A_3 = E_3$. By 3.5, we have $|E_3| = p - 2$, and thus $p = |A_1| + |A_2| + (p - 2) + 1 \ge p + 1$, a contradiction.

Case $m_{12} = m - 1$, i.e. $p = |A_1| + |A_2| + |E_3| - 1$. But now $n \ge m$ is equal to $m_{12} + 1$, and by applying 3.5 to the cycle (I, I_1, I_2) we obtain $|A_1| = |A_2| = 1$ and thus $|E_3| = p - 1$, a contradiction.

Case $m_{12} = m$, i.e. $p = |A_1| + |A_2| + |E_3|$.

If $\varepsilon = 0$, $A_3 = E_3$ is non-empty, and we can interchange I_2 and I_3; for the cycle (I, I_1, I_3) we can discard as above the cases $m_{13} = m \pm 1$, and thus $m_{13} = m$, with again $\varepsilon = 0$, and thus $A_2 = E_2$, and similarly $A_1 = E_1$. We conclude that $p = |E_1| + |E_2| + |E_3| = |E_1| + |E_2| + |E_{12}| + \Delta = p + \Delta$ implies $\Delta = 0$ and thus $(p, m, n) = (3, 6, 7)$. Then the graph of $\{I, I_1, I_2, I_3\}$ is a star of centre I with six indices of weight two, which contradicts Proposition 3.5.

We are left with the case $\varepsilon = 1$, $|A_3| = |E_3| - 1$. Since $n = m + 1$, we have, by 3.5, $n = 3p - 2$ and $|E_i| = p - 2$, and thus $|A_3| = p - 3$; from $n = m_{12} + 1$, we obtain $|A_1| = |A_2| = 1$.

If $(p, n) = (3, 7)$, then $A_3 = \emptyset$ and $I = E_1 \cup E_2 \cup \{7\}$. Let I' be another solution of this type, for instance $I' = E_1 \cup E_3 \cup \{7\}$. Then we can apply Proposition 5.1 to the set $\{I, I', I_1\}$, since $I \cap I' \cap I_1 = E_1$ is not empty. But I,I' and I_1 have indices of weight one (respectively those of E_2, E_3 and E_{12}), a contradiction.

If $p \ge 4$, A_3 is not empty and we may interchange (as above) I_3 with I_1 or I_2, and obtain $|A_3| = 1$, which implies $p = 4$, and $n = 10$. Let $I = A_1 \cup A_2 \cup A_3 \cup \{10\}$ and $I' = A'_1 \cup A'_2 \cup A'_3 \cup \{10\}$ be two distinct solutions, for instance $A_3 \ne A'_3$. If $A_1 = A'_1$, consider as above the set $\{I, I', I_1\}$. It has indices of weight 3 (those of A_1), and also of weight 1 in I ($A_3 \smallsetminus A'_3 = A_3$), in I' (A'_3) and in I_1 ($E_{12} \cup E_{13}$), a contradiction with Proposition 5.1, since $(p,n) \neq (4,8)$. We conclude that for all i, A_i is distinct from A_i', i.e. since E_i has two elements, A_i and A_i' are complementary in E_i. $\square$

Lemma 7.3. *Here we suppose that the $A_{ij} = I \cap E_{ij}$ are not all empty. Then*

(i) $m+1$ does not belong to I;

(ii) there exists a pair (i,j) such that $A_{ij} = E_{ij}$, unique except for $(p,n) = (3,7)$, where $I_0 = E_{12} \cup E_{23} \cup E_{13}$ may belong to $\mathcal{F}$;

(iii) we have, for (i,j,k) permutation of $\{1,2,3\}$,

$$A_{ij} = E_{ij} \Longleftrightarrow A_k = \emptyset\,;$$

(iv) if $A_{12} = E_{12}$ and if $\emptyset \subsetneq A_{13} \subsetneq E_{13}$, then $(A_1, A_{23}) = (E_1, \emptyset)$.

Proof. We suppose for instance $A_{12} \neq \emptyset$. By Proposition 5.1, we know that the graph of $\{I, I_1, I_2\}$ is a path (since $I \cap I_1 \cap I_2 = A_{12} \neq \emptyset$), and that its vertex of valency 2 has no index of weight 1. The sets of indices of weight one in I, I_1 and I_2 are respectively $(I \cap \{m+1\}) \cup A_3$, $(E_1 \smallsetminus A_1) \cup (E_{13} \smallsetminus A_{13})$ and $(E_2 \smallsetminus A_2) \cup (E_{23} \smallsetminus A_{23})$. The sets of indices of weight two in $I \cap I_1$, $I \cap I_2$ and $I_1 \cap I_2$ are respectively $A_1 \cup A_{13}$, $A_2 \cup A_{23}$, and $E_{12} \smallsetminus A_{12}$.

First suppose $A_{12} \neq E_{12}$. Then in the path above I_1 and I_2 are adjacent, and one of them has valency two and thus contains no index of weight one, the other one is not adjacent to I. Thus

$$\emptyset \subsetneq A_{12} \subsetneq E_{12} \Longrightarrow (A_1, A_2, A_{13}, A_{23}) = \begin{matrix} (E_1, \emptyset, E_{13}, \emptyset) \quad \text{or} \\ (\emptyset, E_2, \emptyset, E_{23}) \qquad \end{matrix} ,$$

which establishes the “existence part” of (ii), and (up to exchange of 2 and 3) the item (iv).

Suppose for instance $A_{12} = E_{12}$. Then, we have a path $I_1 \sim I \sim I_2$ (since I_1 and I_2 are no more adjacent), I has no index of weight one: $m+1$ does not belong to I as stated in (i), and A_3 is empty, as stated in the part $\Rightarrow$ of(iii).

Conversely, suppose $A_3 = \emptyset$. If $\emptyset \subsetneq A_{12} \subsetneq E_{12}$, we obtain $I = E_1 \cup E_{13} \cup A_{12} \subset I_1$, thus $I = I_1$, a contradiction. If $A_{12} = \emptyset$, we may suppose (by (ii)) for instance $A_{13} = E_{13}$, and by the part $\Rightarrow$ of (iii), $A_2 = \emptyset$: $I = A_1 \cup A_{23} \cup E_{13}$, with A_1 and $A_{23} \neq \emptyset$ (otherwise, I should be a strict subset or I_3 or I_1). Then, the set $\{I, I_1, I_2\}$ is a cycle, with $m' = |I \cup I_1 \cup I_2| = m - |E_3| \leq m - 1$. By Proposition 3.5, we must have $m' = m-1$ and $n = m'+1$. By 3.5 again, the last equality implies $|I \cap I_1| = 1$, i.e. $|A_1 \cup E_{13}| = 1$, a contradiction. Thus, $A_{12} = E_{12}$, as announced.

It remains to discuss the "unicity" in (ii). Suppose $A_{12} = E_{12}$ and $A_{13} = E_{13}$ for instance. We then have $A_2 = A_3 = \emptyset$, and $I = A_1 \cup E_{12} \cup E_{13} \cup A_{23}$, with $A_{23} \neq \emptyset$ (since $I \not\subset I_1$). By (iv), we conclude $A_{23} = E_{23}$, since A_{12} and A_{13} are non empty. Thus all A_i are empty and I coincides with $E_{12} \cup E_{13} \cup E_{23}$ i.e. the set W_2 of indices of weight 2 in the cycle (I_1, I_2, I_3). Equaling the cardinalities we obtain $p = |W_2| = 3p - m$, thus $m = 2p$ and $(p, n) = (3, 7)$. □

Apart from the two "exotic" solutions for $(p, n) = (3, 7)$ or $(4, 10)$ exhibited in 7.2 and 7.3, we just have proved that the set $\mathcal{F} \smallsetminus \{I_1, I_2, I_3\}$ is a disjoint union of three components $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$, where $\mathcal{F}_i = \{I \in \mathcal{F}, I \neq I_1, I_2, I_3 \mid \ I \cap E_i = \emptyset\}$. We now evaluate their cardinality.

Lemma 7.4. *The set* $\mathcal{F}_3 = \{I \in \mathcal{F}, I \neq I_1, I_2 \mid (A_3, A_{12}) = (\emptyset, E_{12})\}$ *contains at most* $p - |E_{12}| - 1$ *elements.*

Proof. Let
$$I = A_1 \cup A_2 \cup E_{12} \cup A_{13} \cup A_{23}\,,$$
with A_1, A_2, $E_{13} \smallsetminus A_{13}$ and $E_{23} \smallsetminus A_{23}$ non-empty, be an element of $\mathcal{F}_3$. Its intersection $B_1 = A_1 \cup E_{12} \cup A_{13}$ with I_1 contains E_{12} thus is not empty.

We now prove that *w*hen I runs through $\mathcal{F}_3$ the sequence $B_1 = I \cap I_1$ is totally ordered. Let $I' = A'_1 \cup A'_2 \cup E_{12} \cup A'_{13} \cup A'_{23}$ be another element of $\mathcal{F}_3$. Put $B'_1 = I' \cap I_1$, $X = I_1 \smallsetminus B_1$ and $X' = I_1 \smallsetminus B'_1$. We suppose that B_1 and B'_1 satisfy no inclusion. Since they both contain E_{12}, Corollary 5.2 shows that $I_1 = B_1 \cup B'_1$ and that $X \cap X' = \emptyset$. In particular we obtain $E_{13} = A_{13} \cup A'_{13}$ and $A_2 \cap A'_2 = \emptyset$. Since A_{13} and A'_{13} are distinct from E_{13}, the first relation proves that there are not empty, thus by Lemma 7.3, that $A_1 = A'_1 = E_1$. Now we observe that the intersections $A_2 \cup E_{12} \cup A_{23}$ and $A'_2 \cup E_{12} \cup A'_{23}$ of I and I' with I_2, satisfy no inclusion, since A_2 and A'_2 are non-empty and disjoint. By exchanging I_1 and I_2 we deduce that A_1 and A'_1, which both coincide with E_1, must be disjoint, a contradiction. We conclude (by Corollary 5.2) that B_1 and B'_1 satisfy a strict inclusion, for instance $B'_1 \subsetneqq B_1$, i.e.
$$A'_1 \subset A_1 \quad \text{and} \quad A'_{13} \subset A_{13}\,,$$
equality holding in at most one inclusion. In particular, suppose $A'_1 = A_1$, then $A_{13} \supsetneqq A'_{13}$ is not empty, and by the lemma above, $A_1 = E_1$. The totally ordered sequence $(A_1)_{I \in \mathcal{F}_3}$ contains a strictly increasing sequence of non-empty, strict subpaces of E_1 (with at most $|E_1| - 1$ terms) and of at most $|E_{13}|$ terms A_1 equal to E_1 (associated with a strictly increasing sequence of strict subspaces A_{13} of E_{13}). We have then $|\mathcal{F}_3| \leq |E_1| + |E_{13}| - 1 = p - |E_{12}| - 1$ as announced. □

Coming back to the proof of Theorem 7.1, we conclude that the family $\mathcal{F}$ contains apart from I_1, I_2, I_3 at most $3p-3-|W_2| = m-3$ non-exotic terms. The proof is complete for $n \le 3p-3$, i.e. $n = m$.

Case $n = m+1$, *i.e.* $n = 3p-2$. Since $n = m+1$, the E_{ij} are singletons, and their strict subspaces are empty. The elements of $\mathcal{F}_3$ for instance are of the form $I = A_1 \cup A_2 \cup E_{12}$, with A_1 and A_2 non-empty. Actually, we have seen in the above proof that the sequence $(A_1)_{I \in \mathcal{F}_3}$ contains at most $|E_{13}| = 1$ term equal to E_1, so is a strictly totally ordered sequence of non-empty subspaces of E_1, with at most $|E_1| = p-2$ terms. Of course similar remarks are valid for the subspaces A_2 of E_2. We now prove that *i*f two families $\mathcal{F}_i$ are non-empty, one of them at least is a singleton. Let $I = A_1 \cup A_2 \cup E_{12}$ and $I' = A'_1 \cup A'_3 \cup E_{13}$ be elements of $\mathcal{F}_3$ and $\mathcal{F}_2$ respectively. First, A_1 and A'_1 must satisfy an inclusion. Otherwise, (I, I_1, I', I_3, I_2) should be a cycle for the relation $\sim$. Indeed, the sets of indices of weight 2 in $I \cap I_1$, $I' \cap I_1$, $I' \cap I_3$, $I_3 \cap I_2$ and $I \cap I_2$ (respectively $A_1 \smallsetminus A'_1$, $A'_1 \smallsetminus A_1$, $A'_3 \smallsetminus A_3 = A'_3$, $E_{23} \smallsetminus (A_{23} \cup A'_{23}) = E_{23}$ and $A_2 \smallsetminus A'_2 = A_2$), should be all non-empty. Now, since $|I \cup I' \bigcup_j I_j| = |\bigcup_j I_j| = m = n-1$, by 3.5, the subsets above should all be singletons, in particular A_2 and A'_3, implying that A_1 and A'_1 should contains $p-2$ elements, i.e. both coincide with E_1, a contradiction. Therefore we may suppose $A'_1 \subset A_1$. The intersections $B_1 = A_1 \cup E_{12}$ and $B'_1 = A'_1 \cup E_{13}$ of I and I' with I_1 satisfy $B_1 \cap B'_1 = A'_1 \neq \emptyset$, and thus, by Corollary 5.2, $E_1 = A_1 \cup A'_1$, i.e. $A_1 = E_1$, which specifies uniquely $I = E_1 \cup \{a_2\} \cup E_{12}$ in $\mathcal{F}_3$: $|\mathcal{F}_3| = 1$ as announced.

If $\mathcal{F}_1$ is empty, we have $|\bigcup \mathcal{F}_i| \le 1 + (p-2)$, and $\mathcal{F}$ contains at most $p+2$ non-exotic elements.

Otherwise, let $I" = A"_2 \cup A"_3 \cup E_{23}$ be an element of $\mathcal{F}_1$. By exchanging $I"$ with I' or I, we know that $A"_2$ and A_2 on the one hand, $A"_3$ and A'_3 on the other hand, must satisfy an inclusion, and that the larger of the subsets coincides with E_2 or E_3 respectively. We thus have

$$|A_2| = 1 \Rightarrow A"_2 = E_2 \Rightarrow |A"_3| = 1 \Rightarrow A_3 = E_3\,.$$

: $\bigcup \mathcal{F}_i$ contains at most 3 elements, of the form $E_1 \cup \{a_2\} \cup E_{12}$, $\{a'_1\} \cup E_3 \cup E_{13}$ and $E_2 \cup \{a"_3\} \cup E_{23}$, with uniquely determined elements a_i, a'_i or $a"_i$ in E_i.

We conclude that $|\bigcup \mathcal{F}_i| \le \max(3, p-1)$, and thus that in the case $n = 3p-2$, $\mathcal{F}$ contains $p+2$ (resp. 6) non-exotic elements if $p \ge 4$ (resp. $p = 3$).

To complete the proof of the theorem, it remains to discuss the occurrence of the "exotic" elements when $(p, n) = (3, 7)$ or $(4, 10)$. Actually, in both cases, an exotic element, say I, described by Lemma

7.2 is inconsistent with an element, say J, of $\bigcup \mathcal{F}_j$: there exists $k = 1, 2, 3$ such that the set $\{I, J, I_k\}$ contradicts Proposition 5.1. So $\mathcal{F}$ contains at most $6 + 1 = n$ (resp. $6 = p + 2$) elements when $p = 3$ (resp. $p = 4$). This completes the proof of Theorem 7.1.

8. Kissing number of a lattice of index 2 , maximal length

The goal of this section is to prove Theorem 1.1 by giving an explicit upper bound for the number s of pairs $\pm x$ of minimal vectors of the lattice, bound depending on the dimension n modulo 6.

Theorem 8.1. *Let L be a lattice of dimension $n \geq 6$, index 2 with length $\ell = n$. Bounds for the half kissing number s of L are given in the following table.*

n mod 6	*upper bound for s*
0	19 *if* $n = 6$; $(2n^2 + 24n - 45)/9$ *if* $n \geq 12$
1	24 *if* $n = 7$; $(2n^2 + 20n - 13)/9$ *if* $n \geq 13$
2	32 *if* $n = 8$; $(2n^2 + 22n - 25)/9$ *if* $n \geq 14$
3	37 *if* $n = 9$; $(2n^2 + 24n - 54)/9$ *if* $n \geq 15$
4	44 *if* $n = 10$; $(2n^2 + 20n - 4)/9$ *if* $n \geq 16$
5	$(2n^2 + 22n - 34)/9$ $(n \geq 11)$

Proof. To compute the number $s = \frac{|S(L)|}{2}$ of pairs of minimal vectors of the lattice L, we use the following description

$$S(L) = S(L_0) \cup \S_0 \cup S_1 \cup S_2 \cup \cdots \cup S_{\lfloor \frac{n}{2} \rfloor},$$

where $S(L_0)$ stands for the set of minimal vectors of the lattice $L_0 = \langle e_1, e_2, \dots, e_n \rangle$, and S_p for the set of pairs $\pm x$ where x is a minimal vectors of type p. Let $t_p = \frac{|S_p|}{2}$ denote the number of such pairs. Since $S(L_0) = \{\pm e_1, \pm e_2, \cdots \pm e_n\}$ and $S_0 = \{\pm e\}$, we obtain

$$s = n + 1 + \sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} t_p$$

where we shall use of the estimations of the t_p given in the sections above, and the sharper one obtained for $t_1 + t_2$ in 4.2:

$$s \leq n + 10 + \sum_{p=3}^{\lfloor \frac{n}{2} \rfloor} t_p\,.$$

For $p \geq 3$, let T_1 and T_2 denote the bounds for t_p given by Theorems 6.1 and 7.1. If S_p may contain a cycle of length 3, i.e.,by 7.1, if $\frac{n+2}{3} \leq p \leq \frac{n-1}{2}$, we obtain for t_p the estimation $t_p \leq \max(T_1, T_2)$. Otherwise, $t_p \leq T_1$. Now suppose $p = \frac{n+2}{3}$. If $(p, n) = (3, 7)$, T_1 and T_2 coincide with $n = 7$; if $p \geq 4$, $T_2 = p+2 < p+5 \leq T_1$, and again $\max(T_1, T_2) = T_1$. The bound T_2 is to take into account for the integers p such that $\frac{n+2}{3} < p \leq \frac{n-1}{2}$, i.e. for the elements of

$$\mathcal{P} = \{p_1, p_1+1, \ldots, p_k\}, \quad \text{with} \;\; p_1 = \lceil \frac{n+3}{3} \rceil, \; p_k = \lfloor \frac{n-1}{2} \rfloor .$$

Actually, $\mathcal{P}$ is empty for $n = 6$, $n = 8$ and $n = 10$, it contains the type $p = 3$ for $n = 7$ only (and then $T_1 = T_2 = 7$), the type $p = 4$ for $n = 9$ (and then $T_1 = T_2 = 9$). Thus, for $p \in \mathcal{P}$ and $n \leq 10$, we have $t_p \leq T_1 = \max(T_1, T_2)$, while for $p \in \mathcal{P}$ and $n \geq 11$, we have $t_p \leq T_2 = \max(T_1, T_2)$ (since then $T_2 = n \geq \frac{n-1}{2} + 6 \geq p + 6 \geq T_1$).

For $n \leq 10$, we sum up the bounds given by 6.1 5.3 and 4.2:
$n = 6$: $s \leq 7 + 9 + 4 = 20$ (bound to be improved below);
$n = 7$: $s \leq 8 + 9 + 7 = 24$;
$n = 8$: $s \leq 9 + 9 + 8 + 6 = 32$;
$n = 9$: $s \leq 10 + 9 + 8 + 9 = 36$;
$n = 10$: $s \leq 11 + 9 + 8 + 10 + 6 = 44$.

From now on, we suppose $n \geq 11$.

Let Σ_1 denote the sum of the bounds T_1 given by 6.1 for t_p, $3 \leq p \leq \lfloor \frac{n}{2} \rfloor$: $\Sigma_1 = 8 + \sum_{p=4}^{\lfloor \frac{n}{2} \rfloor} (p+6) - \varepsilon = \lfloor \frac{n}{2} \rfloor (\frac{\lfloor \frac{n}{2} \rfloor + 13}{2}) - 16 - \varepsilon$, with $\varepsilon = 1$ if n is odd, $\varepsilon = 5$ if n is even:

$$\Sigma_1 = \frac{n^2 + 24n - 161}{8} \text{ if } n \text{ is odd,} \quad \Sigma_1 = \frac{n^2 + 26n - 168}{8} \text{ if } n \text{ is even}$$

For $p \in \mathcal{P}$ and $n \geq 11$ (thus $p \geq 5$), we must replace the bound T_1 by the bound $T_2 = n$. Let Σ_2 denote the sum of these correcting terms $T_2 - T_1 = n - (p+6)$ (resp. $n - (p+5)$ for $p \neq \frac{n-1}{2}$ (resp. $= \frac{n-1}{2}$). We have

$$\begin{aligned}
\Sigma_2 = & \sum_{p \in \mathcal{P}} (n - 6 - p) + \varepsilon, \text{ with } \varepsilon = 1 \text{ if } n \text{ is odd} \\
= & k(n - p_1 - 6) - 1 - 2 - \cdots - (k-1) + \varepsilon, \text{ with } k = (p_k - p_1 + 1) \\
= & (p_k - p_1 + 1)(n - \frac{p_1 + p_k}{2} - 6) + \varepsilon .
\end{aligned}$$

One easily checks the following expressions of this correcting term, depending on n modulo 6.

n	$72\Sigma_2$
$\equiv 0$	$7n^2 - 114n + 432$
$\equiv 1$	$7n^2 - 128n + 625$
$\equiv 2$	$7n^2 - 130n + 592$
$\equiv 3$	$7n^2 - 96n + 297$
$\equiv 4$	$7n^2 - 146n + 76$
$\equiv 5$	$7n^2 - 112n + 457$

We now use the inequality $s \le n + 1 + 9 + \Sigma_1 + \Sigma_2$ to obtain the table of Theorem 8.1 for $n \ge 7$. Of course the bounds for $s = n + \sum t_p$ obtained by bounding separately the t_p are not optimal. This is the case when $(p, n) = (3, 6)$: the maximal value 4 of t_3 is inconsistent with the maximal value 9 of t_1+t_2, which leads to the bound $s \le 6+1+9+3 = 19$ instead of 20.

[Suppose $t_3 = 4$. The four sets I of type 3 are, up to permutation, $I_1 = \{1,2,3\}$, $I_2 = \{1,3,4\}$, $I_3 = \{1,4,5\}$, $I_4 = \{1,5,2\}$; then, no index $i \in \{1,2,\dots,6\}$ has valency 3 for the relation "$i \equiv j$ if $e - e_i - e_j$ is minimal", as we now prove. There are 3 cases to consider according as the weight of i (with respect to the I_j) is equal to 4, 2 or 0. First, $i = 1$ has at most valency 2: Proposition 3.3 prevents $1 \equiv 6$, and proves that $1 \equiv 2$ is inconsistent with $1 \equiv 3$ or $1 \equiv 5$. For $i = 2$, Proposition 3.3 proves that $2 \equiv 3$ is inconsistent with $2 \equiv 6$, and by Proposition 3.5 we see that $2 \equiv 5$ is not possible, and that $2 \equiv 3$ is inconsistent with $2 \equiv 1$ (for instance the sets $\{2,1\}$, $\{2,3\}$, and $I_2 = \{1,3,4\}$ form a cycle of length 3 and $m = 4$ indices, impossible for $n = 6$). The same argument implies that $6 \equiv 2$ is inconsistent with $6 \equiv 3$ (and $6 \equiv 5$). Thus, by Corollary 4.2, we have $t_1 + t_2 \le 8$.] □

The difference between $\frac{n(n+1)}{2}$ and the bound for s given in 8.1 takes the values

$$2, 4, 4, 9, 11, 16, 19, 26, 30, 36, 44, \dots$$

for $n = 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, \dots,$ is always positive and monotone increasing, and asymptotic to $5n^2/18$ as $n \to \infty$. This completes the proof of Theorem 1.1.

References

[B-R] E.P. Baranovskii, S.S. Ryshkov, *Derivation of perfect lattices from admissible centerings*, Russian Math. Survey 40,4 (1985),155–156.

[C-S] J.H. Conway, N.J.A. Sloane, *Low-Dimensional Lattices. III. Perfect Forms*, Proc. Royal Soc. London **A 418** (1988), 43–80.

[D-S-V] M. Dutour Sikiric, A. Schürmann, F. Vallentin, *Classification of eight dimensional perfect forms*, Math. arXiv, NT/0609388.

[M] J. Martinet, *Bases of minimal vectors in lattices, II*, Arch. Math. 89 (2007), 541–551.