SYMMETRIC GROUPS AND LATTICES

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Abstract. This paper deals with various problems in lattice theory involving local extrema. In particular, we construct infinite series of highly symmetric spherical 3-designs which include some of the examples constructed in [9] in dimensions 5 and 7. We also construct new types of dual-extreme lattices.

Résumé. *Quelques applications de l'algorithme de Voronoï équivariant.* Nous considérons dans cet article divers problèmes de la théorie des réseaux liés à des questions d'extrema locaux. En particulier, nous construisons des séries infinies de 3-designs sphériques qui englobent certains de ceux construits dans [9] en dimensions 5 et 7. Nous construisons également de nouveaux types de réseaux dual-extrêmes.

1. Introduction.

Let E be an n-dimensional Euclidean space equipped with scalar product $x \cdot y$, and let $\mathcal{L}$ be the set of lattices (discrete subgroups of rank n) in E. For a lattice $\Lambda \in \mathcal{L}$, we denote by $\min \Lambda$ its *minimal norm* $\min \Lambda = \min_{x \in \Lambda \smallsetminus \{0\}} x \cdot x$, and by $\det \Lambda$ the determinant of the Gram matrix $(e_i \cdot e_j)$ of any $\mathbb{Z}$-basis $(e_1, e_2, \dots, e_n)$ of Λ. The *density* of the sphere packing associated with Λ is measured by the *Hermite invariant of* Λ

$$\gamma(\Lambda) = \frac{\min \Lambda}{\det \Lambda^{1/n}}\,;$$

another invariant attached to this packing is the *sphere of* Λ, i.e. the set

$$S(\Lambda) = \{x \in \Lambda \mid x \cdot x = \min \Lambda\}$$

of *minimal vectors* of Λ, and its *kissing number* $2s = |S(\Lambda)|$.

A lattice Λ is called *extreme* if it achieves a local maximum of γ over $\mathcal{L}$. In 1907, Voronoi showed that a lattice is extreme if and only if it is both *perfect* and *eutactic*. Geometrical definitions of these notions are as follows; we denote by $\mathrm{End}^s(E)$ the space of symmetric endomorphisms of E and for any non-zero $x \in E$, $p_x \in \mathrm{End}^s(E)$ is the orthogonal projection onto the line $\mathbb{R}x$.

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A lattice Λ is called *perfect* if the p_x, $x \in S(\Lambda)$ span $\mathrm{End}^s(E)$. This notion belongs to linear algebra, as well as the following enlarged notion of eutaxy.

A lattice Λ is *weakly eutactic* if there exists real numbers $\rho_x, x \in S(\Lambda)$ such that $\mathrm{Id}_E = \sum_{x \in S(\Lambda)} \rho_x\, p_x$. We remark that this condition implies that the sphere $S(\Lambda)$ spans the space E, and also that any system of eutaxy coefficients gives way, by averaging, to another such system invariant under the automorphism group $\mathrm{Aut}(\Lambda)$ of Λ. Now the lattice Λ is *eutactic* if it has a system of *strictly positive* eutaxy coefficients; note that this classical notion of eutaxy involves convexity, and coincides when $s = n$ with the weaker one, characterized by the pairwise orthogonality of the n lines $\mathbb{R}\, x, x \in S(\Lambda)$.

While revisiting Voronoi's theory in the setting of spherical designs, Venkov raised a more restrictive notion of eutaxy: the lattice Λ is called *strongly eutactic* if we can choose a system of eutaxy coefficients that are *all equal*. Strong eutaxy amounts to the property of $S(\Lambda)$ to be a spherical 2-*design* (or equivalently 3-*design*, since $S(\Lambda)$ is symmetric); see [10], Prop. 6.2. Empirical observations in low dimension allow us to suspect that strong eutaxy is rare: for example in five dimensions, there are 127 weakly eutactic lattices (the complete enumeration was done by Batut in 2000, [1]), and only eight among them are strongly eutactic. However, no systematic enumeration is so far done beyond dimension 5; in [9], various examples of strongly eutactic lattices are given in dimensions 6 and 7, most of them related to root lattices or cubic lattices.

In Section 4, we construct, from the point of view of group representations, two infinite sequences of strongly eutactic lattices which include some of the "exotic" lattices in the lists above, and provide new ones. In particular, we obtain in all odd dimension $n \geq 5$ two integral lattices B_n and C_n with minimal norms $n-2$ and n respectively.

Another aim of this paper deals with duality, i.e. involves together with the lattice $\Lambda \in \mathcal{L}$ its dual $\Lambda^* = \{x \in E \mid \forall\, y \in \Lambda,\ x \cdot y \in \mathbb{Z}\}$. In 1989, we introduced the *dual-Hermite invariant*

$$\gamma'(\Lambda) = \sqrt{\min \Lambda \, \min \Lambda^*}$$

and characterized *dual-extremality*, i.e. extremality with respect to γ', by the following notions of perfection and eutaxy: Λ is *dual-perfect* if the p_x, $x \in S(\Lambda)$ together with the p_y, $y \in S(\Lambda^*)$ span $\mathrm{End}^s(E)$; Λ is *dual-eutactic* if there exists a relation $\sum_{x \in S(\Lambda)} \rho_x\, p_x = \sum_{y \in S(\Lambda^*)} \rho'_y\, p_y$ with strictly positive coefficients ρ_x, ρ'_y (see [7], Ch. 2 and 3, § 8).

Note that these conditions imply for the kissing numbers of Λ and Λ^* the inequality

$$s+s^* \geq \frac{n(n+1)}{2}+1\,. \tag{1}$$

It is clear that a pair of dual lattices both eutactic and one of them perfect is dual-extreme. At the time [8] was written, all known dual-extreme lattices, to within the single exception of an extreme 7-dimensional lattice whose dual is not eutactic, were obtained using this trick. Hence they were rational, and moreover they satisfied the strongest inequality $s+s^* \geq \dfrac{n(n+1)}{2}+n+1$ for all $n \geq 2$. (Actually, these properties hold in dimensions $2,3,4$.) Examples of dual-extreme pairs of dual lattices, none of them perfect, not both eutactic and with $s+s^* = \dfrac{n(n+1)}{2}+n$ were found for even $n \geq 8$ in [8], then for $n=5$ in [3]. The example of [3] is moreover the first example of an *irrational* dual-extreme lattice (its field of definition is quadratic); a 7-dimensional example, defined over a cubic field, can be obtained in the same way.

We prove in Section 2 that for each even $n \geq 8$, the densest section of the Coxeter lattice $\mathbb{A}_{n+1}^{(n+2)/2}$ is a dual-extreme, n-dimensional lattice which realizes the lower bound of (1); then, (1) is optimal for infinitely many dimensions.

Recall that a lattice Λ is *symplectic* if there is an isometry σ of E such that $\sigma(\Lambda)=\Lambda^*$ and $\sigma^2=-\operatorname{Id}$. Note that such an isoduality gives E a complex structure; thus n must be even. Symplectic lattices are important in connection with complex and algebraic geometry, since they correspond to principally polarized Abelian varieties (see [6]). There is also a "Voronoi theory" for them: one can characterize *symplectic-extreme lattices* (which achieve local maxima of density over all symplectic lattices in E) by suitable notions of *symplectic-perfection* and *symplectic-eutaxy* (see [7], Ch. 11, § 7). Obviously, among symplectic lattices, we have

$$\text{extreme} \implies \text{dual-extreme} \implies \text{symplectic-extreme}\,.$$

We shall display in Section 5 an infinite series of rational non-perfect, symplectic lattices which are nevertheless dual-extreme, that we shall identify with the symplectic-extreme family F_n previously constructed by Bavard in a Riemannian setting ([2]).

The method used is the equivariant version of the Voronoi algorithm established in [5] and outlined in Section 3. Given a finite group G of isometries of E, there are for the G-lattices in E natural notions of

G-perfection and of *G-neighbourhood*. Our examples stem from convenient neighbouring paths connecting G-perfect lattices. The groups G we use are related to the symmetric group S_n acting as a permutation group on an orthonormal basis of E: S_n itself, or some subgroups stabilizing given subspaces. Root lattices play a key role in our examples.

2. A generalization of some Coxeter lattices.

We describe now certain lattices related to the root lattice $\mathbb{A}_n$, using a basis $\mathcal{B}_n = (e_1, e_2, \dots, e_n)$ for its dual $\mathbb{A}_n^*$ with Gram matrix

$$\mathrm{Gram}(\mathcal{B}_n) = \frac{1}{n+1}\begin{pmatrix} n & -1 & -1 & \cdots & -1 \\ -1 & n & -1 & \cdots & -1 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -1 & -1 & -1 & \cdots & n \end{pmatrix} = I_n - \frac{1}{n+1} J_n\,,$$

where I_n and J_n denote respectively the *unit matrix* and he *all-ones matrix*. The dual basis $\mathcal{B}_n^* = (e_i^*)$ affords for $\mathbb{A}_n$ the Gram matrix

$$A_n = I_n + J_n\,.$$

The quotient $\mathbb{A}_n^*/\mathbb{A}_n$ is cyclic of order $n+1$, hence for $n \geq 1$ and r a divisor of $n+1$, there is a unique sublattice of $\mathbb{A}_n^*$ which contains $\mathbb{A}_n$ to index r; it was called $\mathbb{A}_n^r$ by Coxeter in 1951. The dual of $\mathbb{A}_n^r$ is $\mathbb{A}_n^{\frac{n+1}{r}}$. The Coxeter lattices are stable under the group $\mathrm{Aut}(\mathbb{A}_n) \simeq \{\pm\,\mathrm{Id}\} \times S_{n+1}$.

In this section we focus on the extreme odd-dimensional lattice $\Lambda = \mathbb{A}_n^{(n+1)/2}$ $(n = 5, 7\dots)$. In order to define the densest (resp. the isodual) hyperplane sections of Λ, we look for vectors of norm 2 (resp. $\frac{n+1}{4}$) in the dual lattice $\Lambda^* = \mathbb{A}_n^2$. They constitute a single orbit under the action of $\mathrm{Aut}\,\Lambda$, for which we may choose e_n^* (resp. $\frac{e_1^* - e_2^* + \cdots - e_{n-1}^* + e_n^*}{2}$) as a representative. Taking the sections of Λ orthogonal to these vectors, we define two canonical series of even-dimensional lattices, isometric only in dimension 6.

Definition 2.1. *For $n \geq 4$ the n-dimensional generalized Coxeter lattice Cox_n is $\mathbb{A}_n^{(n+1)/2}$ if n is odd, and the section of $\mathbb{A}_{n+1}^{(n+2)/2}$ orthogonal to e_{n+1}^* if n is even. Moreover, when n is even, we denote by Cox'_n the section of $\mathbb{A}_{n+1}^{(n+2)/2}$ orthogonal to $\frac{e_1^* - e_2^* + \cdots + e_n^* - e_{n+1}^*}{2}$.*

More explicitly, for $n \geq 4$ even:

$$\begin{cases} \mathrm{Cox}_n = \Big\{ \sum_{i=1}^{n} x_i e_i \;\Big|\; (x_i) \in \mathbb{Z}^n \text{ and } \sum_i x_i \equiv 0 \mod 2 \Big\} \\ \mathrm{Cox}'_n = \Big\{ \sum_{i=1}^{n+1} x_i e_i \;\Big|\; (x_i) \in \mathbb{Z}^{n+1} \text{ and } \sum_{i=1}^{n+1} (-1)^i x_i = 0 \Big\}, \end{cases}$$

where $(e_1, \ldots, e_{n+1})$ is a basis for $\mathbb{A}_{n+1}^*$ whose Gram matrix is $I_{n+1} - \frac{1}{n+2} J_{n+1}$.

The lattice Cox'_n is not the densest section of $\mathbb{A}_{n+1}^{\frac{n+2}{2}}$, except in the cases $n = 4$ and $n = 6$, where $\mathrm{Cox}'_4 \simeq \mathbb{A}_2 \bigotimes \mathbb{A}_2$ and $\mathrm{Cox}'_6 \simeq \mathrm{Cox}_6 \simeq \mathbb{D}_6^+$. It has nevertheless interesting properties of duality that we shall discuss in Section 5 by inserting Cox'_n in an equivariant family of symplectic, n-dimensional lattices.

The remaining of the present section is devoted to the lattice Cox_n, n even, which except for $n = 4$ is the densest section of $\mathbb{A}_{n+1}^{\frac{n+2}{2}}$. The lattice $\mathrm{Cox}_6 \simeq \mathbb{D}_6^+$ has exceptional duality and symmetries. For $n = 8$ onwards, the sequence Cox_n acquires a certain regularity.

Proposition 2.2. *Let $n \geq 8$ be even. The lattice Cox_n and its dual have minimal norms $\frac{2n}{n+2}$ and $\frac{3}{2}$ respectively, and spheres*

$$S(\mathrm{Cox}_n) = \{\pm(e_i + e_j),\ 1 \leq i < j \leq n\,;\ \pm(e_1 + \cdots + e_n)\},$$

$$S(\mathrm{Cox}_n^*) = \{\pm e'_i, 1 \leq i \leq n\},$$

where $(e'_1, \ldots, e'_n)$ is the basis dual to $(e_1, \ldots, e_n)$. The automorphism group $\mathrm{Aut}(\mathrm{Cox}_n)$ is isomorphic to $\{\pm \mathrm{Id}\} \times S_n$, acts transitively on $S(\mathrm{Cox}_n^*)$, and has two orbits on $S(\mathrm{Cox}_n)$, with one and $\frac{n(n-1)}{2}$ pairs $(\pm x)$ of vectors respectively.

Proof. The minimal vectors of $\mathbb{A}_{n+1}^{\frac{n+2}{2}}$, namely $\pm(e_i + e_j)$, $0 \leq i < j \leq n+1$ where $e_0 = -(e_1 + e_2 + \cdots + e_{n+1})$, belong to $\mathbb{R}\,\mathrm{Cox}_n$ if and only if $1 \leq i < j \leq n$ or $(i,j) = (0, n+1)$. The sphere and minimum of its section Cox_n follow immediately.

The basis (e'_i) has Gram matrix $(I_n - \frac{1}{n+2} J_n)^{-1} = I_n + \frac{1}{2} J_n$ and associated quadratic form

$$Q(x_i) = \sum x_i^2 + \tfrac{1}{2} \left(\sum x_i\right)^2.$$

To determine the minimal norm of

$$\mathrm{Cox}_n^* = \langle e'_1, \ldots, e'_n, \frac{e'_1 + \cdots + e'_n}{2} \rangle,$$

we must evaluate Q over half-integers congruent modulo $\mathbb{Z}$. It takes values $\geq \frac{3}{2}$ on $\mathbb{Z}^n$, with equality only on the canonical basis (up to signs); if the x_i are halves of odd integers, we have $\sum x_i^2 \geq \frac{n}{4} > \frac{3}{2}$ since $n \geq 8$. So, the minimum and sphere of Cox_n^* are as stated in 2.2.

Definition 2.1 shows that Cox_n is stable under the symmetric group S_n permuting $\{e_1, \dots, e_n\}$. It remains to prove that the group $\{\pm \mathrm{Id}\} \times S_n$, whose action on $S(\mathrm{Cox}_n)$ is as stated in 2.2, is for $n \geq 8$ the full automorphism group of Cox_n. Since $S(\mathrm{Cox}_n^*) = \{\pm e_i'\}$ has rank n, $\mathrm{Aut}(\mathrm{Cox}_n) = \mathrm{Aut}(\mathrm{Cox}_n^*)$ is generated by some permutations and sign changes of the e_i'. Actually, the scalar products $e_i' \cdot e_j'$ displayed in $\mathrm{Gram}(e_i')$ allow all permutations of the e_i', but just the negation of all of them, which completes the proof of 2.2. □

Note that the statements in 2.2 about the minimum of Cox_n, its sphere and the action of S_n on it still hold in dimension 6. In contrast, the dual lattice Cox_6^* has exceptional minimal vectors, namely all permutations of $\frac{e_1'+e_2'+e_3'-e_4'-e_5'-e_6'}{2}$. We conclude that the group S_6 fixes one minimal vector of Cox_6, but no minimal vector of its dual: *the similarities* $\mathrm{Cox}_6 \simeq \mathrm{Cox}_6^*$ *do not preserve the action of* S_6.

We now consider the extremal properties of the lattices Cox_n. For the sake of completeness, we state the results for all n, even or odd. However, the results are known for n odd: the lattices Cox_n and their duals are strongly eutactic (because their group acts irreducibly), and moreover perfect except for Cox_5^*. Let now $n \geq 8$ be even. Then, the lattices $L = \mathrm{Cox}_n$ and L^* are not perfect (because $s(L^*) < s(L) < \frac{n(n+1)}{2}$), L^* is not eutactic (its n pairs of minimal vectors are not pairwise orthogonal), and the proof of the theorem below will show that L is not strongly eutactic.

Theorem 2.3. *For all $n \geq 7$, the lattices* Cox_n *are eutactic and dual-extreme.*

Proof. We only need to consider even dimensions $n \geq 8$. Eutaxy and dual-perfection involve the orthogonal projections onto the minimal vectors of $L = \mathrm{Cox}_n$ and L^*, namely the projections p_0 onto $e_1+\cdots+e_n$, $p_{i,j}$ onto $e_i + e_j$, $1 \leq i < j \leq n$, and p_i' onto e_i', $1 \leq i \leq n$ respectively.

For the proof of the dual-extremality, we shall establish that there exists up to multiplication by a positive scalar a unique non-trivial linear relation $f = 0$, where f has the form

$$a\, p_0 + \sum_{1 \leq i < j \leq n} a_{i,j}\, p_{i,j} - \sum_{1 \leq h \leq n} a_h'\, p_h' \,,$$

and that all its coefficients a, $a_{i,j}$, and a'_h are strictly positive.
The matrix $M = (f(e'_i).e'_j)$ has entries

$$m_{i,j} = \begin{cases} \dfrac{1}{\mu}(a + a_{i,j}) - \dfrac{1}{\mu^*}(\dfrac{a'_i + a'_j}{2} + \dfrac{\sum a'_k}{4}) & \text{if } i \neq j\,, \\ \dfrac{1}{\mu}(a + \displaystyle\sum_h a_{i,h}) - \dfrac{1}{\mu^*}(2a_i\prime + \dfrac{\sum a'_k}{4}) & \text{if } i = j\,, \end{cases}$$

where $\mu = \frac{2n}{n+2}$ and $\mu^* = \frac{3}{2}$ denote the minimal norms of L and L^*, and where we consider that the $a_{i,j}$ are defined for all $i \neq j$, $i,j \leq n$, and are symmetric in i, j. For any fixed $i \neq j$ we have
$\sum_h(m_{i,h} - m_{j,h}) - 2m_{i,i} + 2m_{j,j} = \frac{n-6}{2\mu^*}(a'_j - a'_i)$; so, if $n \neq 6$ the coefficients a'_i in any relation $f = 0$ do not depend on i; it readily follows that there is, up to scale, a *unique* relation between the projections p_x, $x \in S(L) \cup S(L^*)$, which reads

$$n(n^2 + 2n - 12)\, p_0 + 4n \sum_{i<j} p_{i,j} = 3(n^2 - 4) \sum_i p'_i\,.$$

Note that taking into account the action of S_n on both $S(L)$ and $S(L^*)$, since the space of S_n-lattices has dimension 2 – see Section 4 below– the existence of such a relation was evident; for dual-eutaxy the only problem was that of the signs of its coefficients. The same remark applies for the eutaxy relation $\mathrm{Id} = ap_0 + b\sum p_{i,j}$. In matrix form, it reads $\mu(I_n + \frac{1}{2}J_n) = (n-2)bI_n + (a+b)J_n$. That leads to the unique system of eutaxy coefficients

$$(a,b) = (\tfrac{n(n-4)}{n^2-4}, \tfrac{2n}{n^2-4})\,,$$

which are strictly positive, but distinct for $n \geq 7$. □

As mentioned in the introduction, *the pair* $(\mathrm{Cox}_n, \mathrm{Cox}_n^*)$ *realizes in all even dimension* $n \geq 8$ *the lower bound* $s + s^* = \frac{n(n+1)}{2} + 1$ *for the kissing numbers of a dual-extreme pair.*

Note that the eutaxy relation above for Cox_n is still valid in dimensions 4 and 6: the lattice Cox_4 is semi-eutactic (the only one up to dimension 4), and the lattice $\mathrm{Cox}_6 \simeq \mathbb{D}_6^+$ strongly eutactic (its group $W(\mathbb{D}_6)$ acts irreducibly). Note also that the proof of 2.3 shows that for any $n \geq 4$ the dimension of the span in $\mathrm{End}^s(E)$ of the projections p_x, $x \in S(L)$ is equal to $s = \frac{n(n-1)}{2} + 1$. But for $n = 4$ or $n = 6$, the projections onto the minimal vectors of the dual belong to this span, which excludes dual-extremality.

3. Equivariant theory of lattices.

Let $G \subset \mathrm{O}(E)$ be a finite group of isometries for which E affords a rational representation. The set $\mathcal{L}_G$ of G-stable lattices in E consists of orbits under the action of the centralizer of G in the linear group $\mathrm{GL}(E)$. These orbits are in one-to-one correspondence with the integral equivalence classes of integral representations of G. For the equivariant Voronoi theory we replace $\mathrm{End}^s(E)$ by the commuting subspace

$$\mathrm{End}^s_G(E) = \{f \in \mathrm{End}^s(E) \mid \forall \sigma \in G, f\sigma = \sigma f\}$$

of G, and the orthogonal projections p_x by their means

$$\omega_x = \frac{1}{|G|} \sum_{\sigma \in G} p_{\sigma x} \,.$$

A lattice $\Lambda \in \mathcal{L}_G$ is said in [4] to be *G-perfect* if the ω_x, $x \in S(\Lambda)$ span $\mathrm{End}^s_G(E)$. This notion arises in studying the density of G-lattices: a G-lattice Λ is *G-extreme*, i.e. achieves a local maximum of γ over $\mathcal{L}_G$, if and only if it is both G-perfect and eutactic, see [7], Chapters 3 and 11.

We described in the joint paper [5] with Sigrist a *Voronoi algorithm with group G*, obtaining a connected *Voronoi G-graph* which classifies all similarity classes of G-perfect lattices affording a given *integral* representation of G; see also [7], Ch. 13. From now on we rather adopt the point of view of quadratic forms. In the space $\mathrm{Sym}_n(\mathbb{R})$ of $n \times n$ symmetric real matrices equipped with the scalar product $\langle M, N\rangle = \mathrm{Tr}(M\,N)$, a lattice $L \in \mathcal{L}$ is represented by one of its Gram matrices A. Its minimal norm is then the minimum $\min A$ of the quadratic form $A[x] = xA{}^t x$ $(= \langle A, {}^t xx\rangle)$ for integral vectors $x = (x_1, x_2, \dots, x_n) \neq 0 \in \mathbb{Z}^n$, and its sphere is represented in $\mathbb{Z}^n$ by $S(A) = \{x \neq 0, x \in \mathbb{Z}^n \mid A[x] = \min A\}$.

Now, given an integral representation $\phi : G \to \mathrm{GL}_n(\mathbb{Z})$ of G, we denote by $\mathcal{S}_\phi$ the space of *ϕ-invariant* matrices, i.e. matrices $A \in \mathrm{Sym}_n(\mathbb{R})$ such that for all $\sigma \in G$, ${}^t\phi_\sigma A \phi_\sigma = A$, and by $\mathcal{P}_\phi$ the subset of positive definite ϕ-invariant matrices. To a vector $x \in \mathbb{R}^n$ we attach the matrix $\Omega_x := \frac{1}{|G|} \sum_{\sigma \in G} \phi_\sigma {}^t xx\, {}^t\phi_\sigma$, which is invariant under the dual representation of G. By definition, the *G-Voronoi domain* $\mathcal{D}_A$ of a matrix $A \in \mathcal{P}_\phi$ is the convex hull, in the dual space $\mathcal{S}^*_\phi$, of the half-lines generated by the $\Omega_x, x \in S(A)$. We can now translate in matrix form the notions above for G-lattices, keeping the same terminology: a eutaxy relation reads $A^{-1} = \sum_{x \in G\backslash S(A)} \rho_x \Omega_x$, and A is G-perfect if the dimension of its domain is maximal, i.e. equal to

$$N_G = \dim(\mathcal{S}_\phi) \quad (= \dim(\mathrm{End}^s_G(E)))\,.$$

We now describe the neighbouring of a G-perfect matrix $A \in \mathcal{S}_\phi$. Its domain $\mathcal{D}_A$ is bounded by a certain set of $(N_G - 1)$-dimensional faces, or *facets*. (Of course, a G-perfect matrix has at least N_G orbits of pairs $(\pm x)$ of minimal vectors. When it has just N_G orbits, we easily obtain each facet of its domain by simply removing one of the orbits.) To each facet there corresponds in $\mathcal{S}_\phi$ a uniquely determined *Voronoi path* orthogonal to it that we now describe. Let $\mathcal{F}$ be a facet and $F \neq 0 \in \mathcal{S}_\phi$ orthogonal to $\mathcal{F}$ and oriented towards the interior of $\mathcal{D}_A$. Then the open Voronoi path through $\mathcal{F}$ is

$$c_\mathcal{F} : \quad t \in (0, \theta) \mapsto A_t = A + t\,F\,,$$

where $\theta \in (0, \infty]$ is defined by $\theta = \sup\{t > 0 \mid \min A_t = \min A\}$. We can alternatively define the path $c_\mathcal{F}$ as the set of matrices $M \in \mathcal{S}_\phi$ such that $\min M = \min A$ and $S(M) = S_\mathcal{F}$ with

$$S_\mathcal{F} = \{x \in S(A) \mid \Omega_x \in \mathcal{F}\}\,.$$

Following [7], Ch. 13, Th. 3.5., there are two cases:
• If $S_\mathcal{F}$ spans $\mathbb{R}^n$, θ is finite and the matrix A_θ is G-perfect again, with sphere $S(A_\theta) \supsetneq S_\mathcal{F}$; we say that A and A_θ are *G-neighbours*. Their domains $\mathcal{D}_A$ and $\mathcal{D}_{A_\theta}$ have just the face $\mathcal{F}$ in common.
• If $S_\mathcal{F}$ spans a strict subspace of $\mathbb{R}^n$, θ is infinite. We say that the path is a *dead-end*.

We summarize now some results of the classification of eutactic lattices applied to equivariant Voronoi paths (for the general setting, see [7] or [1]).

Proposition 3.1. *Let $c_\mathcal{F}$ be an equivariant Voronoi path. Then:*

1. *There is at most one weakly eutactic matrix in $c_\mathcal{F}$.*
2. *If $c_\mathcal{F}$ is a dead-end, it contains no weakly eutactic matrix.*
3. *A matrix $M \in c_\mathcal{F}$ is weakly eutactic if and only if it minimizes Hermite's function γ over $c_\mathcal{F}$.*
4. *If $c_\mathcal{F}$ connects two G-extreme matrices, it contains a weakly eutactic matrix.*

The dimension N_G of Voronoi's space depends on the canonical decomposition of the representation E (for precise formulae, see [7], Ch. 11, Th. 3.8.). In this paper, we only consider direct sums $E = E_1 \perp \cdots \perp E_k$ of k non-isomorphic $\mathbb{R}$-irreducible representations of real type, and then we have $N_G = k$. Note that the irreducible case is trivial for Voronoi's theory: there are only finitely many similarity classes of G-lattices. In contrast, the dimension $N_G = 2$ is of special interest to effortlessly build 3-designs:

Proposition 3.2. *If $N_G = 2$, any weakly eutactic matrix in a G-Voronoi path is strongly eutactic. In particular, there is a strongly eutactic matrix associated with any pair (A, B) of G-extreme neighbours.*

Proof. Let S be the (common) sphere of the matrices in the path: the span of the Ω_x, $x \in S$, which contains the positive matrix $X = \sum_{x\in S} {}^t x\, x$, also contains the inverse of any weakly eutactic M in the path. From the definition of a Voronoi path, this span has dimension $N_G - 1$: under our hypothesis it is a line, hence any eutaxy relation reads $M^{-1} = \rho X$ for some ρ necessarily positive: this is strong eutaxy. □

We conclude this section by describing the classification of a family of G-lattices (i.e. an orbit in $\mathcal{L}_G$ under the action of the centralizer of G in $\mathrm{GL}(E)$) in the case $N_G = 2$. Denote by E_1 and E_2 the two irreducible (non isomorphic) components of E.

Theorem 3.3. *If $N_G = 2$, the Voronoi graphs have the form*

$$\infty_1 \text{——} L_1 \text{——} \cdots \text{——} L_r \text{——} \infty_2\,,$$

where $L_1, \dots, L_r$ are G-perfect lattices, and where the dead-ends ∞_1 and ∞_2 correspond to the orthogonal projections onto the irreducible components E_1 and E_2. Moreover, $r = 1$ if and only if E_1 and E_2 both contain minimal vectors of L_1.

Proof. We follow [7], Ch. 13, § 4: in the dual space $\mathcal{S}_\phi^*$, the set of positive matrices is the angular domain with angle $\pi/2$ and edges (Ω_1) and (Ω_2) associated with the projections p_1 and p_2 onto E_1 and E_2. The Voronoi domains of the G-perfect matrices form a partition of this right angle into angular domains. We now prove that there are only finitely many such domains. For this proof we adopt the geometric point of view, and evaluate angles in the space $\mathrm{End}^s(E)$ equipped with the scalar product $\langle f, g\rangle = \mathrm{Tr}(f \circ g)$. Recall that G-perfect lattices are proportional to integral lattices, and that up to similarity, there are finitely many of them (see [7], Ch. 11, § 9, 9.1 and 9.5). Let $\mathcal{E}$ denote the (infinite) set of primitive integral copies of the G-perfect lattices. Their minimal norms take only finitely many values, let m be the biggest. Then the inner products $x \cdot y$, $x, y \in S(L)$, bounded by $\frac{m}{2}$, take only finitely many integral values when L runs through $\mathcal{E}$. Hence there are finitely many values of $\langle p_x, p_y\rangle = \frac{(x\cdot y)^2}{(x\cdot x)(y\cdot y)}$, $x, y \in S(L)$, $L \in \mathcal{E}$; by averaging over G we obtain the same conclusion for the $\langle \omega_x, \omega_y\rangle$, and thus for the cosine $\frac{\langle \omega_x,\omega_y\rangle}{\sqrt{\langle \omega_x,\omega_x\rangle\langle \omega_y,\omega_y\rangle}}$ of the angle of Ω_x and Ω_y in $\mathcal{S}_\phi^*$. So, the partition of the angular domain (Ω_1, Ω_2) into Voronoi domains is finite, and so is the Voronoi graph (here we do not discard G-lattices isometric to lattices already found). Finally the first and last lattices in the sequence must have minimum vectors in the irreducible subspaces E_1 and E_2 respectively, corresponding the edges Ω_1 and Ω_2 of their respective domains. □

The proof above shows that either the L_i are pairwise non–G-isometric, or the graph is symmetric. Examples of both types occur in the next section (the S_n-graph for $\mathbb{A}_n$ is symmetric if $n = 2$, but not if $n \geq 3$), which also provides graphs with $r = 1$, 2 or 3.

4. Two sequences of strongly eutactic lattices.

Throughout this section, the group S_n is viewed as the permutation group of a given orthonormal basis $\mathcal{B} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ of E.

There are exactly two irreducible stable subspaces of E under this action of S_n, namely the line $\langle \sum_i \varepsilon_i \rangle$ and its orthogonal complement: the space of G-invariant symmetric bilinear forms on E has dimension $N_{S_n} = 2$. In matrix form with respect to the basis $\mathcal{B}$, it is the span

$$\mathcal{S} = \{aI_n + bJ_n,\ (a, b) \in \mathbb{R}^2\}$$

of I_n and of the all-ones matrix J_n.

Any irreducible root lattice in E admits at least one group action associated with the above permutation representation of S_n. Actually, one can prove that, up to $\mathbb{Z}[S_n]$-isomorphism, there is just one such action, except in the case of $\mathbb{D}_4$ and $\mathbb{E}_8$, which have two.

The S_n-family of $\mathbb{D}_n$. We represent the lattice

$$\mathbb{D}_n = \Big\{ \sum_i x_i \varepsilon_i \mid (x_i) \in \mathbb{Z}^n,\ \sum_i x_i \text{ even } \Big\}$$

by the generator matrix G_n and the Gram matrix $D_n = {}^t\!G_n\, G_n$:

$$G_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & (-1)^{n-1} \end{pmatrix}, \quad D_n = \begin{pmatrix} 2 & 1 & \cdots & 0 & 1 \\ 1 & 2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 2 & (-1)^{n-1} \\ 1 & 0 & \cdots & (-1)^{n-1} & 2 \end{pmatrix}$$

of its basis $(\varepsilon_1 + \varepsilon_2,\ \varepsilon_2 + \varepsilon_3,\ \cdots,\ \varepsilon_{n-1} + \varepsilon_n,\ (-1)^{n-1}\varepsilon_n + \varepsilon_1)$. This leads to the following description.

Proposition 4.1. *A lattice in E affords the same integral representation of S_n as $\mathbb{D}_n$ if and only if it is isometric to one with Gram matrix*

$$[a, b] := aD_n + bK_n, \ \text{ with } a > 0 \ \text{ and } \ a + nb > 0\,,$$

where the matrix $K_n = {}^t\!G_n\, J_n\, G_n$ is equal to $4J_n$ if n is odd and to $\begin{pmatrix} 4J_{n-1} & \mathbf{0} \\ {}^t\mathbf{0} & 0 \end{pmatrix}$ if n is even. It has determinant $\det([a, b]) = 4a^{n-1}(a + nb)$.

Proof. The Gram matrix M of a basis permuted by S_n has constant non-diagonal entries, say b, and constant diagonal entries, say $a+b$, hence $M = aI_n + bJ_n$. The signature and determinant of M follow from its characteristic equation $(x-a)^{n-1}(x-(a+nb))$. Conjugating by the matrix G_n with determinant 2 completes the proof. $\square$

In the following graphs, the S_n-perfect lattices are represented by their Gram matrices $[a,b] = aD_n + bK_n$ scaled to minimum 2. The symbol ∞ indicates a dead-end.

Proposition 4.2. *For $n \geq 4$, the Voronoi graph associated with the representation $\phi_{\mathbb{D}_n}$ of S_n is:*

$$n \text{ even}: \quad \infty \;—\; \underset{[1,0]}{\overset{\mathbb{D}_n}{\bullet}} \;———\; \underset{[\frac{n+2}{n},\frac{-1}{n}]}{\overset{\mathrm{Cox}_n}{\bullet}} \;—\; \infty\,;$$

$$n \text{ odd}: \quad \infty \;—\; \underset{[1,0]}{\overset{\mathbb{D}_n}{\bullet}} \;———\; \underset{[\frac{n+1}{n-1},\frac{-1}{n-1}]}{\overset{\mathrm{Cox}_n}{\bullet}} \;———\; \underset{[\frac{3n^2+2n-1}{2n^2-2n},\frac{-3n-1}{2n^2-2n}]}{\overset{N_n}{\bullet}} \;—\; \infty\,,$$

where N_n (n odd) is a S_n-perfect lattice with $s = n+1$ pairs of minimal vectors.

Proof. We have to evaluate the quadratic forms $(aI_n + bJ_n)[(x_i)]$ on the set

$$\mathcal{E} = \{(x_i) \in \mathbb{Z}^n \text{ with } \sum x_i \equiv 0 \mod 2\}$$

where the group S_n acts by permuting the x_i. In this point of view, the matrix I_n represents the lattice $\mathbb{D}_n$; it has minimum 2 on $\mathcal{E}$ attained at two orbits (here, as elsewhere, we waive the distinction between minimal vectors and their negatives): the orbit of $(1,-1,0,\dots,0)$, that we disregard because of Proposition 3.1, and the orbit $\mathcal{O}$ of $(1,1,0,\dots,0)$. The corresponding path from D_n is then $\{{}^t G_n\, M(t)\, G_n\}$, where

$$M(t) = (1+2t)I_n - tJ_n, \quad 0 < t < \theta_n, \quad \text{with } \theta_n = \frac{1}{2\lfloor\frac{n}{2}\rfloor}\,.$$

Indeed, the matrix $2I_n - J_n$ vanishes at $\mathcal{O}$ and the matrix ${}^t G_n\, M(\theta_n)\, G_n$ is S_n-extreme, for as we now prove, it is a Gram matrix for the lattice Cox_n rescaled to minimal norm 2: when n is odd, $\frac{n-1}{n+1}M(\theta_n) = A_n^{-1}$ is the standard Gram matrix for $\mathbb{A}_n^*$, and thus represents on $\mathcal{E}$ the Coxeter lattice $\mathbb{A}_n^{\frac{n+1}{2}}$, of norm $\frac{2(n-1)}{n+1}$; the case where n is even follows from Definition 2.1. The second orbit $\mathcal{O}'$ of norm 2 vectors of $M(\theta_n)$ is that of $(1,1,\dots,1)$ when n is even (and by 3.1 the algorithm terminates), and that of $(1,\dots,1,0)$ when n is odd. In this case, it is easy to prove that for $(a_n,b_n) = (\frac{(3n-1)(n+1)}{2n(n-1)}, -\frac{3n+1}{2n(n-1)})$, the quadratic form $(a_nI_n + b_nJ_n)[x] = a_n \sum (x_i - \frac{\sum x_i}{n})^2 + \frac{1}{2}(\frac{\sum x_i}{n})^2$ has minimum 2 on the set $\mathcal{E}$ with in addition to $\mathcal{O}'$ two minimal vectors $\pm(2,2,\dots,2)$. We conclude that the matrix $[a_n, b_n]$ is S_n-perfect, and that the algorithm terminates at it. $\square$

Note that the family defined by $\mathbb{D}_n$ always contains the Barnes-Coxeter lattice $\mathbb{A}_n^{\frac{n+1}{2}}$: for n even it lies in the dead-end issuing from Cox_n.

We can now enumerate the eutactic lattices in the family.

For $n \geq 7$ odd, the path $(\mathrm{Cox}_n - N_n)$ affords no such lattices. [Dimension five is exceptional, for then the representations defined by $\mathbb{D}_5$ and its dual are integrally equivalent, and afford two eutactic lattices, namely N_5 (class e_6 in [1]) and $\mathbb{D}_5^*$. Following 3.1 (3), we now determine the less dense lattice in the path $\mathbb{D}_n - \mathrm{Cox}_n$. The typical Gram matrix $[1+2t, -t]$, $0 < t < \theta_n$ for this path has determinant $4(1+2t)^{n-1}(1+(2-n)t)$, whose derivative vanishes at

$$t_n = \frac{1}{2n-4} \in (0, \theta_n)$$

for $n > 4$. We can now state the following, where it is to be understood that lattices are defined up to similarity.

Theorem 4.3. *For $n \geq 5$ the strongly eutactic lattice C_n attached to the pair $(\mathbb{D}_n, \mathrm{Cox}_n)$ has Gram matrix $\frac{n-1}{n-2} D_n - \frac{1}{2n-4} K_n$, $s = \frac{n(n-1)}{2}$ pairs of minimal vectors, and automorphism group $\{\pm \mathrm{Id}\} \times S_n$ except for $n = 5$, where $C_5 \simeq \mathbb{A}_5^2$. Its primitive integral copy has minimal norm $n-2$ if n is odd, $2(n-2)$ if n is even.* $\square$

Examples. We display below Gram matrices for C_6 and C_7:

$$C_6 : \begin{pmatrix} 8 & 3 & -2 & -2 & -2 & 5 \\ 3 & 8 & 3 & -2 & -2 & 0 \\ -2 & 3 & 8 & 3 & -2 & 0 \\ -2 & -2 & 3 & 8 & 3 & 0 \\ -2 & -2 & -2 & 3 & 8 & -5 \\ 5 & 0 & 0 & 0 & -5 & 10 \end{pmatrix} ; \qquad C_7 : \begin{pmatrix} 5 & 2 & -1 & -1 & -1 & -1 & 2 \\ 2 & 5 & 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & 5 & 2 & -1 & -1 & -1 \\ -1 & -1 & 2 & 5 & 2 & -1 & -1 \\ -1 & -1 & -1 & 2 & 5 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 & 5 & 2 \\ 2 & -1 & -1 & -1 & -1 & 2 & 5 \end{pmatrix} .$$

The lattice C_6 must be added to the list of 19 displayed in [9], whereas C_7 was discovered by Batut and Sigrist (see [9], p.122) using the cyclic group of order 7 (which does not account for its strong eutaxy).

Other representations of S_n related to root lattices. The integral representations of S_n defined (up to integral equivalence) by the lattice $\mathbb{A}_n$, $n \geq 2$, or by the lattice $\mathbb{D}_n^*$, $n \geq 6$, introduce no new 3-design (in each case there is just one pair of S_n-extreme lattices, with associated eutactic lattice $\mathbb{Z}^n$ and $\mathbb{D}_n^*$ respectively). Note that the $\mathbb{D}_4^*$ construction affords the second representation for $\mathbb{D}_4 \simeq \mathbb{D}_4^*$, for which it has no neighbour (the graph is $\infty - \mathbb{D}_4 - \infty$).

The representation of S_6 defined by the root lattice $\mathbb{E}_6$ affords the trivial Voronoi graph $\infty - \mathbb{E}_6 - \infty$.

For $n \geq 8$ even, the (maybe not integral) Kneser-neighbour L of the cubic lattice $\bigoplus \mathbb{Z}\varepsilon_i$ has (even for $n = 8$) exactly two $\mathbb{Z}[S_n]$-structures corresponding to its two constructions $\mathbb{D}_n^+ \simeq \mathbb{D}_n^-$ consisting in adjoining to $\mathbb{D}_n$ the vectors $\frac{\varepsilon_1+\cdots++\varepsilon_{n-1}+\varepsilon_n}{2}$ and $\frac{\varepsilon_1+\cdots+\varepsilon_{n-1}-\varepsilon_n}{2}$ respectively. Dimension 8 is of no interest from our point of view: as $\mathbb{D}_8^+$ the lattice $L = \mathbb{E}_8$ has no S_8-neighbour, and as $\mathbb{D}_8^-$ it has just one neighbour, non-eutactic, with $s = 9$. For $n \geq 10$, Voronoi's graphs have a uniform structure. In particular, they introduce two sequences of strongly eutactic lattices, say C_n^+, $n \geq 18$ and C_n^-, $n \geq 12$, with $s = s^* = \frac{n(n-1)}{2}$; both lattices contain the lattice C_n to index 2, have the same sphere as it, and double density. The S_n-neighbour of $\mathbb{D}_n^-$, N_n^- say, is perfect in the classical sense (with just $s = \frac{n(n+1)}{2}$ pairs of minimal vectors), and even extreme for $n \geq 12$. It belongs to families of *perfect lattices with possible odd minimum* of the form

$$\Lambda = \langle e_1, \dots, e_n, e = \frac{e_1 + \cdots + e_n}{d} \rangle,\ n \geq 2d\,,$$

constructed by the second author. In the example above, the primitive integral copy of the lattice N_n^- has odd minimum if and only if $n \equiv 2 \mod 4$. In particular, it affords the first example of a perfect, integral 10-dimensional lattice having an odd minimum (namely 11).

The lattice B_n, n odd. The lattices C_n, $n \geq 5$ described in Theorem 4.3 are not nested. In contrast, in odd dimension, strong eutaxy happens to be preserved by the following section of codimension two. Here we suppose that E has odd dimension $2m+1$, $m \geq 3$. We consider the subspace E' of dimension $2m-1$

$$E' = \Big\{ \sum_{i=1}^{2m+1} x_i\varepsilon_i \ \Big|\ x_1 = 0 \ \text{and}\ \sum_{k=1}^{m} x_{2k} = \sum_{k=1}^{m} x_{2k+1} \Big\}$$

and its stabilizer G in S_{2m+1}

$$G \simeq S_m \times S_m \rtimes C_2\,,$$

where C_2 is generated by the product of transpositions

$$\textstyle\prod_{i\geq 2, i \text{ even}} (i, 2m+3-i).$$

The corresponding equivariant families again have dimension $N_G = 2$. In particular, the family of G-lattices defined by the root lattice $\mathbb{A}_{2m-1} = \mathbb{D}_{2m+1} \cap E'$ consists of the sections by E' of the $(2m+1)$-dimensional lattices described in Proposition 4.1. The typical invariant matrix can be obtained by removing from the $(2m+1) \times (2m+1)$ matrices $[a, b]$ the first and last rows and columns; it has the form

$aA'_{2m-1} + 4bJ_{2m-1}$, where

$$A'_n = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot\cdot & \\ 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix};$$

its determinant is $\det(aA'_{2m-1} + 4bJ_{2m-1}) = 2m\, a^{2m-2}(a + 2mb)$.

Proposition 4.4. *Let $n = 2m - 1 \geq 5$ be an odd integer. The n-dimensional lattice $B_n = C_{n+2} \cap E'$ with Gram matrix $\frac{n+1}{n} A'_n - \frac{2}{n} J_n$ is strongly eutactic. Its automorphism group $\simeq \{\pm \mathrm{Id}\} \times G$ acts transitively on its m^2 pairs of minimal vectors, and its primitive integral copy has minimal norm n.*

Proof. It is easy to prove that the family of G-lattices defined by $\mathbb{A}_n$ has Voronoi graph

$$\infty \;\text{——}\; \overset{\mathbb{A}_n}{\bullet} \;\text{—————}\; \overset{\mathrm{Cox}_{n+2} \cap E'}{\bullet} \;\text{——}\; \infty\,,$$

where $\mathrm{Cox}_{n+2} \cap E'$ is G-extreme with two G-orbits of minimal vectors. (Its two eutaxy coefficients $\frac{1}{m+1}$ and $\frac{m-1}{2m+2}$ are distinct except in dimension 5, where $\mathbb{E}_7^* \cap E' \simeq \mathbb{A}_5^2$.) The typical matrix for the Voronoi path is $(1 + 2t)A'_n - 4tJ_n$, with $0 \leq t \leq \theta_{n+2} = \frac{1}{n+1}$ (see the proof of Theorem 4.3). The derivative of its determinant

$$(n + 1)(1 + 2t)^{n-1}(1 + (1 - n)t)$$

again vanishes at $t = t_{n+2} = \frac{1}{2n} \in (0, \frac{1}{n-1})$: the strongly eutactic lattice attached to the pair $(\mathbb{A}_n, \mathrm{Cox}_{n+2} \cap E')$ coincides with $C_n \cap E'$. □

The Gram matrix of B_5 obtained from that of C_7 by omitting the first and last rows and columns is equivalent to the matrix given by Batut (see [9], p. 121).

5. A family of symplectic lattices in dimension $2m \geq 6$.

Here again E is a Euclidean space of odd dimension $2m+1$ ($m \geq 3$) equipped with an orthogonal basis ($\varepsilon_i, 1 \leq i \leq 2m+1$) acted on by the symmetric group S_{2m+1}. We consider its hyperplane

$$F = \Big\{ \sum_i x_i \varepsilon_i \;\Big|\; \sum_{k=1}^{m} x_{2k} = \sum_{k=0}^{m} x_{2k+1} \Big\}\,,$$

and the stabilizer of F in S_{2m+1}

$$G \simeq S_{m+1} \times S_m\,,$$

where S_{m+1} acts on $\{\varepsilon_{2k+1}\}$ and S_m on $\{\varepsilon_{2k}\}$.

Regarded as a representation of G, F has three irreducible components, and any family of G-lattices in F has dimension $N_G = 3$. As an example, let us describe the one defined by the root lattice $\mathbb{A}_{2m}$, leaving the proof to the reader.

Proposition 5.1. *A $(2m)-$dimensional lattice is a G-lattice in the family of $\mathbb{A}_{2m}$ if and only if it is similar to one with Gram matrix*

$$[a,b,c] := \begin{pmatrix} \scriptstyle (a-b)I_m+bJ_m & \scriptstyle cI_m \\ \scriptstyle cI_m & \scriptstyle cA_m \end{pmatrix},$$

$$0 < c < \min\{a-b, (m+1)a + (m^2-1)b\}\,.$$

In particular, $[2,0,1]$ *is the Gram matrix of the basis*

$$(\varepsilon_{2i} + \varepsilon_{2i+1}, \quad \varepsilon_{2i+1} - \varepsilon_1), \quad 1 \le i \le m$$

for $\mathbb{A}_{2m}$. □

Note that the path $\{[2, 2-2c, c],\ c \in [1, 1+\frac{1}{m}]\}$, which connects the lattices $\mathbb{D}_{2m+1} \cap F = \mathbb{A}_{2m}$ and $\sqrt{\frac{m+1}{m}}\,\mathrm{Cox}_{2m+1} \cap F = \sqrt{\frac{m+1}{m}}\,\mathrm{Cox}'_{2m}$, is not a G-Voronoi path.

From the classification provided by Voronoi's algorithm we only retain that there are just four G-perfect matrices (all of them G-extreme): apart from $[2,0,1]$, there is a not too interesting matrix (minimum 2, $s = 2m+1$), and two neighbouring matrices $F_{2m} = [2, \frac{-2}{m}, 1]$ and $F'_{2m} = [2, \frac{-2}{m}, 1 + \frac{2}{m}]$. The remaining of the section is devoted to the path $F_{2m} - F'_{2m}$, namely the set of matrices

$$M(x) := \Big[2, -\frac{2}{m}, \frac{2m+2}{m}x\Big], \quad x_0 \le x \le 1 - x_0\,,$$

where $x_0 = \frac{m}{2m+2}$, which possess additional symmetries.

Note that $M(\frac{1}{2}) = [2, \frac{-2}{m}, 1 + \frac{1}{m}]$ is a Gram matrix for the lattice $\sqrt{\frac{m+1}{m}}\,\mathrm{Cox}'_{2m}$.

To describe group actions, we consider for any x a basis $(e_i,\ 1 \le i \le 2m+1)$ of E acted on by G as (ε_i) (all permutations of the e_{2k} and all permutations of the e_{2k+1}), with the following inner products: $e_i \cdot e_j$ is equal to $-\frac{1}{2m}$ for odd $i-j$, to $2x + \frac{1}{2m}$ (resp. $2(1-x) + \frac{1}{2m}$) for odd (resp. even) $i = j$, and to $-\frac{2x}{m} + \frac{1}{2m}$ (resp. $-\frac{2(1-x)}{m} + \frac{1}{2m}$) for odd (resp. even) $i \ne j$. The basis $(f_1, \dots, f_{2m})$ of F defined by

$$f_i = e_{2i} + e_{2i+1}, \quad f_{i+m} = e_{2i+1} - e_1, \quad 1 \le i \le m$$

has then Gram matrix $M(x)$.

Now for any x, consider the set

$$V = \{e_0, e_1, \dots, e_{2m}, e_{2m+1}\} \quad \text{with} \quad e_0 = -(e_1 + e_2 + \cdots e_{2m+1})\,.$$

The inner products $e_i \cdot e_j$ of these $2m+2$ vectors, for $i = j$ or for $i \neq j$, only depend on i and j modulo 2. Thus, in the group of any matrix $M(x)$ there is a subgroup

$$G' \simeq S_{m+1} \times S_{m+1}\,,$$

where the factors act on the e_{2k} and on the e_{2k+1} respectively. Since F has just two non-isomorphic irreducible G'-subspaces (the spans $\langle e_i - e_j,\, i \text{ and } j \text{ odd}\,\rangle$ and $\langle e_i - e_j,\, i \text{ and } j \text{ even}\,\rangle$), any family of G'-invariant matrices has dimension $N_{G'} = 2$.
Eventually, the product of transpositions $\Pi = \prod_{k=0}^{k=m}(e_{2k}, e_{2k+1}) \in S_{2m+2}$ induces an integral equivalence between the matrices $M(x)$ and $M(1-x)$, which stabilizes $M(\frac{1}{2})$ and connects $F_{2m} = M(x_0)$ and $F'_{2m} = M(1-x_0)$. Therefore, the stabilizer in $\mathrm{GL}_{2m}(\mathbb{Z})$ of the path $\{M(x)\}$ contains a group

$$G'' \simeq <\Pi> \times S_{m+1} \times S_{m+1}\,,$$

and so does the automorphism group of $M(\frac{1}{2})$. *Since the representation F of G'' is irreducible, the matrix $M(\frac{1}{2})$ is strongly eutactic.*

We can now state the following proposition.

Proposition 5.2. (1) *The matrices $M(x) = [2, -\frac{2}{m}, \frac{2m+2}{m}x]$ are up to scale symplectic for any $x \in (0,1)$.*
(2) *For $x \in [x_0, 1-x_0]$, they have minimum* 2.
(3) *The matrices $F_{2m} = M(x_0)$ and $F'_{2m} = M(1-x_0)$ are G'-extreme, with associated strongly eutactic matrix $M(\frac{1}{2})$.*
(4) *For $x \in [x_0, 1-x_0]$, the automorphism group of $M(x)$ is*

$$\mathrm{Aut}(M(x)) = \begin{cases} \{\pm\,\mathrm{Id}\} \times G' & \text{if } x \neq \dfrac{1}{2}, \\[2ex] \{\pm\,\mathrm{Id}\} \times G'' & \text{if } x = \dfrac{1}{2}, \text{ and } m \neq 3. \end{cases}$$

Proof. For assertion (1), put $y = \sqrt{x(1-x)}$. Then

$$\frac{m}{2(m+1)y} M(x) = \frac{1}{y}\begin{pmatrix} A_m^{-1} & xI_m \\ xI_m & (x^2+y^2)A_m \end{pmatrix}$$

is the symplectic matrix canonically identified with the point $(x+iy)A_m$ in the Siegel space $\mathbb{H}_m = \{X + iY \mid X, Y \in \mathrm{Sym}_m(\mathbb{R}), Y > 0\}$ (see [6]). Actually, it lies in the *hyperbolic family* $\{zA_m, z \in \mathbb{H}_1\}$ attached by Bavard to the lattice $\mathbb{A}_m$ ([2]). The determination of the minimum of $M(x)$ is based on the following result of Bavard.

Lemma 5.3. ([2], th. 2.1) *The matrix F_{2m} has minimum 2 and $\frac{(m+1)(3m+2)}{2}$ pairs of minimal vectors.* □

These vectors are divided under G'-action into two orbits that we describe (up to signs) with respect to the set $V = \{e_0, e_1, \dots, e_{2m+1}\}$ corresponding to $x = x_0$:

$$\mathcal{O}_1 = \{e_i + e_j,\, i - j \text{ odd }\}, \quad \mathcal{O}_2 = \{e_i - e_j,\, i < j,\, i \text{ and } j \text{ odd }\},$$

which satisfy the unique eutaxy relation

$$(m+2)(m+1)\,\mathrm{Id} = 2m \sum_{x \in \mathcal{O}_1} p_x + 4 \sum_{x \in \mathcal{O}_2} p_x\,.$$

Clearly, Lemma 5.3 is also valid for $F'_{2m} \sim F_{2m}$, but we have to replace $\mathcal{O}_2$ by the orbit $\mathcal{O}'_2 = \{e_i - e_j,\, i < j,\, i \text{ and } j \text{ even}\}$. From Voronoi's theory it follows that F_{2m} and F'_{2m} are G'-extreme and that, for $x_0 < x < 1 - x_0$, $M(x)$ has minimum 2 and sphere $S(M(x)) = \mathcal{O}_1$. The strong eutaxy of $M(\frac{1}{2})$ completes the proof of assertions (2) and (3). Note that under G-action, $\mathcal{O}_2$ is stable, but $\mathcal{O}_1$ is divided into two G-orbits, and so is $\mathcal{O}'_2$. We conclude that F_{2m} and F'_{2m} are both G-extreme, but that their equivalence does not preserve group action.

For assertion (4), note that any automorphism of $M(x)$ must stabilize the orbit $\mathcal{O}_1$, and thus the matrix $M(\frac{1}{2})$. Taking into account the inclusions we already know, it remains to establish that $\{\pm\,\mathrm{Id}\} \times G''$ is the full automorphism group of $M(\frac{1}{2})$. We leave the somewhat tedious proof to the reader. □

In geometric language, the properties of the matrix $M(\frac{1}{2})$ read:

Corollary 5.4. *For $m \geq 2$ the lattice Cox'_{2m} is symplectic, strongly eutactic, with minimum $\frac{2m}{m+1}$ and $s = (m+1)^2$ pairs of minimal vectors. Its automorphism group is $\{\pm\,\mathrm{Id}\} \times G''$ except for $m = 3$ where $\mathrm{Aut}(\mathbb{D}_6^+) = W(\mathbb{D}_6)$ contains $\{\pm\,\mathrm{Id}\} \times G''$ to index 10.* □

Now let us discuss the extremal properties of the isodual lattice $\boldsymbol{F}_{2m}$ with Gram matrix $F_{2m} = \begin{pmatrix} \frac{2(m+1)}{m} A_m^{-1} & I_m \\ I_m & A_m \end{pmatrix}$.

Theorem 5.5. *For $m \geq 2$ the lattice $\boldsymbol{F}_{2m}$ is dual-extreme, and non-perfect except in the case $m = 2$ where it is isometric to $\mathbb{D}_4$.*

Proof. Since $\Lambda = \boldsymbol{F}_{2m}$ is eutactic and isodual, it is dual-eutactic. Let $\sigma : \Lambda \mapsto \Lambda^*$ be the symplectic similarity corresponding to the matrix $\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$.

To prove that Λ is dual-perfect, we extract from the spheres $S(\Lambda)$ and $S(\Lambda^*) = \sigma(S(\Lambda))$ the following set Σ with $|\Sigma| = \frac{2m(2m+1)}{2}$ elements:
- $e_i + e_j$, $i - j$ odd , $(i,j) \neq (0,1)$;
- $e_i - e_j$, i and $j \geq 3$ odd ;
- $\sigma(e_{2k+1} - e_{2k'+1}) = \frac{m}{m+2}(e_{2k'} - e_{2k})$.

In matrix form with respect to the basis $(e_k,\ 1 \le k \le 2m+1)$ of E, it is easy to prove that the orthogonal projections $p_x, x \in \Sigma$ are linearly independent. Similarly, one can prove that a relation $\sum_{x\in S(\Lambda)} \lambda_x p_x = 0$ has infinitely many solutions depending on m parameters. The rank of the $s = \frac{(3m+2)(m+1)}{2}$ projections is then $s - m < \frac{2m(2m+1)}{2}$ except when $m = 2$. $\square$

6. Tables.

Tables 6.1 and 6.2 display the main invariants of the n-dimensional lattices constructed in this paper. The symbols μ, γ, γ', s, s^*, and g denote respectively the minimum of the primitive integral copy of the lattice L, its Hermite and dual-Hermite invariants, the half-kissing-number of L, of L^*, and the order of the automorphism group. The markers $(*)$ and $(**)$ point out the usual exceptions in low dimension: $(*)$ corresponds to $n = 5$ when $C_5 \simeq A_5^2$, and $(**)$ to $n = 6$ when $\mathrm{Cox}_6 \simeq \mathrm{Cox}'_6 \simeq \mathbb{D}_6^+$.

Table 6.1 *Strongly eutactic lattices of dimension $n \ge 5$*

Lattice	μ	γ	γ'	$s,\ s^*$	g
C_n	$n-2$ if n is odd $2(n-2)$ if n is even	$\left(\frac{2n-4}{n-1}\right)^{\frac{n-1}{n}}$	$\sqrt{2}$	$\frac{n^2-n}{2},\ n^{(*)}$	$2\,n!^{(*)}$
B_n n odd	n	$n\left(\frac{2}{n+1}\right)^{\frac{n+1}{n}}$	$\sqrt{\frac{2n}{n+1}}$	$(\frac{n+1}{2})^2,\ n+1$	$(2\,(\frac{n+1}{2})!)^2$
Cox'_n n even	n if $n\equiv 2 \mod 4$ $2n$ if $n\equiv 0 \mod 4$	$\frac{2n}{n+2}$	$\gamma'=\gamma$	$s=s^*=(\frac{n}{2}+1)^2$	$4\,((\frac{n}{2}+1)!)^{2\,(**)}$

Table 6.2. *Dual-extreme lattices of even dimension $n \ge 6$*

Lattice or matrix	μ	γ	γ'	$s,\ s^*$	g
F_n n even ≥ 6	$\frac{n}{2}$ if $n\equiv 0 \mod 4$ n if $n\equiv 2 \mod 4$	$\sqrt{\frac{4n}{n+4}}$	$\gamma'=\gamma$	$s=s^*=\frac{3n^2+10n+8}{8}$	$2((\frac{n}{2}+1)!)^2$
Cox_n n even ≥ 6	n if $n\equiv 0 \mod 4$ $\frac{n}{2}$ if $n\equiv 2 \mod 4$	$\frac{n\ 2^{\frac{n-3}{n}}}{(n+2)^{\frac{n-1}{n}}}$	$\sqrt{\frac{3n}{n+2}}$	$\frac{n^2-n+2}{2},\ n^{(**)}$	$2\,n!^{(**)}$

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