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SYMPLECTIC LATTICES

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Introduction

The title refers to lattices arising from principally polarized Abelian varieties, which are naturally endowed with a structure of *symplectic* $\mathbb{Z}$-modules. The density of sphere packings associated to these lattices was used by Buser and Sarnak [B-S] to locate the Jacobians in the space of Abelian varieties. During the last five years, this paper stimulated further investigations on density of symplectic lattices, or more generally of *isodual lattices* (lattices that are isometric to their duals, [C-S2]).

Isoduality also occurs in the setting of modular forms: Quebbemann introduced in [Q1] the *modular lattices*, which are integral and similar to their duals, and thus can be rescaled so as to become isodual. The search for modular lattices with the highest Hermite invariant permitted by the theory of modular forms is now a very active area in geometry of numbers, which led to the discovery of some symplectic lattices of high density.

In this survey, we shall focus on isoduality, pointing out its different aspects in connection with various domains of mathematics such as Riemann surfaces, modular forms and algebraic number theory.

1. Basic definitions

1.1 Invariants. Let E be an n-dimensional real Euclidean vector space, equipped with scalar product $x.y$, and let Λ be a lattice in E (discrete subgroup of rank n). We denote by $m(\Lambda)$ its *minimum* $m(\Lambda) = \min_{x\neq 0\in\Lambda} x.x$, and by $\det\Lambda$ the determinant of the Gram matrix $(e_i.e_j)$ of any $\mathbb{Z}$-basis $(e_1, e_2, \cdots e_n)$ of Λ. The density of the sphere packing associated to Λ is measured by the *Hermite invariant* of Λ

$$\gamma(\Lambda) = \frac{m(\Lambda)}{\det\Lambda^{1/n}}.$$

The *Hermite constant* $\gamma_n = \sup_{\Lambda\subset E}\gamma(\Lambda)$ is known for $n \leq 8$. For large n, Minkowski gave linear estimations for γ_n, see [C-S1], I,1.

Another classical invariant attached to the sphere packing of Λ is its kissing number $2s = |S(\Lambda)|$ where

$$S(\Lambda) = \{x \in \Lambda \mid x.x = m(\Lambda)\}$$

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is the set of *minimal vectors* of Λ.

1.2 Isodualities. The *dual lattice of* Λ is

$$\Lambda^* = \{y \in E \mid x.y \in \mathbb{Z} \text{ for all } x \in \Lambda\}.$$

An *isoduality* of Λ is an isometry σ of Λ onto its dual; actually, σ exchanges Λ and Λ^* (since ${}^t\sigma = \sigma^{-1}$), and σ^2 is an automorphism of Λ. We can express this property by introducing the group $\operatorname{Aut}^\# \Lambda$ of the isometries of E mapping Λ onto Λ or Λ^*. When Λ is isodual, the index $[\operatorname{Aut}^\# \Lambda : \operatorname{Aut} \Lambda]$ is equal to 2 except in the *unimodular* case, i.e. when $\Lambda = \Lambda^*$, and the isodualities of Λ are in one-to-one correspondence with its automorphisms.

We attach to any isoduality σ of Λ the bilinear form

$$B_\sigma : (x, y) \mapsto x.\sigma(y),$$

which is integral on $\Lambda \times \Lambda$ and has discriminant $\pm 1 = \det \sigma$.

Two cases are of special interest:

(i) *The form B_σ is symmetric*, or equivalently $\sigma^2 = 1$. Such an isoduality is called *orthogonal*. For a prescribed signature (p, q), $p + q = n$, it is easily checked that the set of isometry classes of σ-isodual lattices of E is of dimension pq. We recover, when $\sigma = \pm 1$, the finiteness of the set of unimodular n-dimensional lattices.

(ii) *The form B_σ is alternating*, i.e $\sigma^2 = -1$. Such an isoduality, which only occurs in even dimension, is called *symplectic*. Up to isometry, the family of symplectic $2g$-dimensional lattices has dimension $g(g + 1)$ (see the next section); for instance, every two-dimensional lattice of determinant 1 is symplectic (take for σ a planar rotation of order 4). Note that an isodual lattice can be both symplectic and orthogonal. For example, it occurs for any 2-dimensional lattice with $s \geq 2$. The densest 4-dimensional lattice $\mathbb{D}_4$, suitably rescaled, has, together with symplectic isodualities (see below), orthogonal isodualities of every indefinite signature.

2. Symplectic lattices and Abelian varieties

2.1 Let us recall how symplectic lattices arise naturally from the theory of complex tori. Let V be a complex vector space of dimension g, and let Λ be a full lattice of V. The complex torus V/Λ is an *Abelian variety* if and only if there exists a *polarization on* Λ, i. e. a positive definite Hermitian form H for which the alternating form $\operatorname{Im} H$ is integral on $\Lambda \times \Lambda$. In the $2g$-dimensional real space V equipped with the scalar product $x.y = \operatorname{Re} H(x, y) = \operatorname{Im} H(ix, y)$, multiplication by i is an isometry of square -1 that maps the lattice Λ onto a sublattice of Λ^* of index $\det(\operatorname{Im} H)$ $(= \det \Lambda)$. This is an isoduality for Λ if and only if $\det(\operatorname{Im} H) = 1$. The polarization H is then said *principal*.

Conversely, let $(E, .)$ be again a real Euclidean vector space, Λ a lattice of E with a symplectic isoduality σ as defined in subsection 1.2. Then E can be made into a complex vector space by letting $ix = \sigma(x)$. Now the real alternating form $B_\sigma(x, y) = x.\sigma(y)$ attached to σ in 1.2(ii) satisfies $B_\sigma(ix, iy) = B_\sigma(x, y)$ (since σ is an isometry) and thus gives rise to the definite positive Hermitian form $H(x, y) = B_\sigma(ix, y) + iB_\sigma(x, y) = x.y + ix.\sigma(y)$, which is a principal polarization for Λ (by 1.2 (ii)).

So, *there is a one-to-one correspondence between symplectic lattices and principally polarized complex Abelian varieties.*

REMARK. In general, if $(V/\Lambda, H)$ is any polarized abelian variety, one can find in V a lattice Λ' containing Λ such that $(V/\Lambda', H)$ is a principally polarized abelian variety. For example, let us consider the Coxeter description of the densest six-dimensional lattice $\mathbb{E}_6$. Let $\mathcal{E} = \{a + \omega b \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$, with $\omega = \frac{-1+i\sqrt{3}}{2}$ be the Eisenstein ring. In the space $V = \mathbb{C}^3$ equipped with the Hermitian inner product $H((\lambda_i), (\mu_i)) = 2\sum \lambda_i \overline{\mu_i}$, the lattice $\mathcal{E}^3 \cup (\mathcal{E}^3 + \frac{1}{1-\omega}(1,1,1))$ is isometric to $\mathbb{E}_6$, and the lattice $\frac{1}{\omega - \overline{\omega}}\{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{E}^3 \mid \lambda_1 + \lambda_2 + \lambda_3 \equiv 0 \quad (1-\omega)\}$ to its dual $\mathbb{E}_6^*$ (see [M]). The rescaled lattice $\Lambda = 3^{\frac{1}{4}}\mathbb{E}_6^*$ satisfies $i\Lambda \subset 3^{-\frac{1}{4}}\mathcal{E}^3 \subset \Lambda^*$: while the polarization H is not principal for Λ, it is principal on $\Lambda' = 3^{-\frac{1}{4}}\mathcal{E}^3$, and the principally polarized abelian variety $(\mathbb{C}^3/\Lambda', H)$ is isomorphic to the direct product of three copies of the curve $y^2 = x^3 - 1$.

2.2 We now make explicit (from the point of view of geometry of numbers) the standard parametrization of symplectic lattices by the *Siegel upper half-space*

$$\mathfrak{H}_g = \{X + iY, \quad X \text{ and } Y \text{ real symmetric } g \times g \text{ matrices}, Y > 0\}.$$

Let $\Lambda \subset E$ be a $2g$-dimensional lattice with a symplectic isoduality σ. It possesses a *symplectic basis* $\mathcal{B} = (e_1, e_2, \cdots, e_{2g})$, i.e. such that the matrix $(e_i.\sigma(e_j))$ has the form

$$J = \begin{pmatrix} O & I_g \\ -I_g & O \end{pmatrix},$$

(see for instance [M-H], p. 7). This amounts to saying that the Gram matrix $A := (e_i.e_j)$ is symplectic. More generally, a $2g \times 2g$ real matrix M is *symplectic* if ${}^tMJM = J$.

We give E the complex structure defined by $ix = -\sigma(x)$, and we write $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, with $\mathcal{B}_1 = (e_1, \cdots, e_g)$. With respect to the $\mathbb{C}$-basis $\mathcal{B}_1$ of E, the generator matrix of the basis $\mathcal{B}$ of Λ has the form $(\,I_g\ Z\,)$, where $Z = X + iY$ is a $g \times g$ complex matrix. The isometry $-\sigma$ maps the real span F of $\mathcal{B}_1$ onto its orthogonal complement $F^\perp$, and the $\mathbb{R}$-basis $\mathcal{B}_1$ onto the dual-basis of the orthogonal projection $p(\mathcal{B}_2)$ of $\mathcal{B}_2$ onto $F^\perp$. Since $Y = \operatorname{Re} Z$ is the generator matrix of $p(\mathcal{B}_2)$ with respect to the basis $(-\sigma)(\mathcal{B}_1) = (p(\mathcal{B}_2))^*$, we have $Y = \operatorname{Gram}(p(\mathcal{B}_2)) = (\operatorname{Gram}(\mathcal{B}_1))^{-1}$; the matrix Y is then symmetric, and moreover Y^{-1} represents the polarization H in the $\mathbb{C}$-basis $\mathcal{B}_1$ of E (since $H(e_h, e_j) = e_h.e_j + ie_h.\sigma(e_j) = e_h.e_j$ for $1 \le h, j \le g$). Now, the Gram matrix of the basis $\mathcal{B}_0 = \mathcal{B}_1 \perp p(\mathcal{B}_2)$ of E is $\operatorname{Gram}(\mathcal{B}_0) = \begin{pmatrix} Y^{-1} & O \\ O & Y \end{pmatrix}$. Since the (real) generator matrix of the basis $\mathcal{B}$ with respect to $\mathcal{B}_0$ is $P = \begin{pmatrix} I_g & X \\ O & I_g \end{pmatrix}$, we have $A = \operatorname{Gram}(\mathcal{B}) = {}^tP \operatorname{Gram}(\mathcal{B}_0) P$, and it follows from the condition "A symplectic" that the matrix X also is symmetric, so we conclude

$$A = \begin{pmatrix} I_g & O \\ X & I_g \end{pmatrix} \begin{pmatrix} Y^{-1} & O \\ O & Y \end{pmatrix} \begin{pmatrix} I_g & X \\ O & I_g \end{pmatrix}, \text{ with } X + iY \in \mathfrak{H}_g.$$

On the other hand, such a matrix A is obviously positive definite, symmetric and symplectic.

Changing the symplectic basis means replacing A by ${}^tPAP$, with P in the symplectic modular group

$$\operatorname{Sp}_{2g}(\mathbb{Z}) = \{P \in \operatorname{SL}_{2g}(\mathbb{Z}) \mid {}^tPJP = J\}.$$

One can check that the corresponding action of $P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ on $\mathfrak{H}_g$ is the homography $Z \mapsto Z' = (\delta Z + \gamma)(\alpha Z + \beta)^{-1}$.

Most of the well known lattices in low even dimension are proportional to symplectic lattices, with the noticeable exception of the above-mentioned $\mathbb{E}_6$: the roots lattices $\mathbb{A}_2$, $\mathbb{D}_4$ and $\mathbb{E}_8$, the Barnes lattice P_6, the Coxeter-Todd lattice K_{12}, the Barnes-Wall lattice BW_{16}, the Leech lattice Λ_{24}... In Appendix 2 to [B-S], Conway and Sloane give some explicit representations $X + iY \in \mathfrak{H}_g$ of them. A more systematic use of such a parametrization is dealt with in section 6.

3. Jacobians

The Jacobian $\operatorname{Jac} C$ of a curve C of genus g is a complex torus of dimension g wich carries a canonical principal polarization, and then the corresponding *period lattice* is symplectic. Investigating the special properties of the Jacobians among the general principally polarized Abelian varieties, Buser and Sarnak proved that, while the linear Minkowski lower-bound for the Hermite constant γ_{2g} still applies to the general symplectic lattices, the general linear upper bound is to be replaced, for period lattices, by a logarithmic one (for explicit values, see [B-S], p. 29), and thus one does not expect large-dimensional symplectic lattices of high density to be Jacobians. The first example of this obstruction being effective is the Leech lattice. A more conclusive argument in low dimension involves the centralizer $\operatorname{Aut}_\sigma(\Lambda)$ of the isoduality σ in the automorphism group of the σ-symplectic lattice Λ: if Λ corresponds to a curve C of genus g, we must have, from Torelli's and Hurwitz's theorems, $|\operatorname{Aut}_\sigma(\Lambda)| = |\operatorname{Aut}(\operatorname{Jac} C)| \leq 2|\operatorname{Aut} C| \leq 2 \times 84(g-1)$. Calculations by Conway and Sloane (in [B-S], Appendix 2) showed that $|\operatorname{Aut}_\sigma(\Lambda)|$ is one hundred times over this bound in the case of the lattice E_8, and one million in the case of the Leech lattice!

However, up to genus 3, almost all principally polarized abelian varieties are Jacobians, so it is no wonder if the known symplectic lattices of dimension $2g \leq 6$ correspond to Jacobians of curves: the lattices $\mathbb{A}_2$, $\mathbb{D}_4$ and the Barnes lattice P_6 are the respective period lattices for the curves $y^2 = x^3 - 1$, $y^2 = x^5 - x$ and the Klein curve $xy^3 + yz^3 + zx^3 = 0$ (see [B-S], Appendix 1). The Fermat quartic $x^4 + y^4 + z^4 = 0$ gives rise to the lattice $\mathbb{D}_6^+$ (the family $\mathbb{D}_{2g}^+$ is discussed in section 7), slightly less dense, with $\gamma = 1.5$, than the Barnes lattice P_6 ($\gamma = 1.512...$) but with a lot of symmetries ($\operatorname{Aut}_\sigma(\Lambda)$ has index 120 in the full goup of automorphisms.

The present record for six dimensions ($\gamma = 1.577...$) was established in [C-S2] by the Conway-Sloane lattice $M(\mathbb{E}_6)$ (see section 7) defined over $\mathbb{Q}(\sqrt{3})$. This lattice was shown in [Bav1], and independently in [Qi], to be associated to the exceptional Wiman curve $y^3 = x^4 - 1$ (the unique non-hyperelliptic curve with an automorphism of order $4g$, viewed in [Qi] as the most symmetric Picard curve).

In the recent paper [Be-S], Bernstein and Sloane discussed the period lattice associated to the hyperelliptic curve $y^2 = x^{2g+2} - 1$, and proved it to have the form $L_{2g} = M_g \perp M_g'$, where M_g is a g-dimensional isodual lattice, and M_g' a copy of its dual. Here the interesting lattice is the summand M_g (its density is that of L_{2g}, and its group has only index 2): it turns out to be, for $g \leq 3$, the densest isodual packing in g dimensions.

Remark. The Hermite problem is part of a more general systole problem (see [Bav1]). So far, although a compact Riemann surface is determined by its polarized

Jacobian, no connection between its systole and the Hermite invariant of the period lattice seems to be known.

4. Modular lattices

4.1 Definition. Let Λ be an n-dimensional *integral* lattice (i.e. $\Lambda \subset \Lambda^*$), which is similar to its dual. If σ is a similarity such that $\sigma(\Lambda^*) = \Lambda$, its norm ℓ (σ multiplies squared lengths by ℓ) is an integer which does not depend on the choice of σ. After Quebbeman, we call Λ a *modular lattice of level.* Note that level one corresponds to unimodular lattices.

For a given pair (n, ℓ), the (hypothetical) modular lattices have a prescribed determinant $\ell^{n/2}$, thus, up to isometry, there are only finitely many of them; as usual we are looking for the largest possible minimum m (the Hermite invariant $\gamma = \frac{m}{\sqrt{\ell}}$ depends only on it). In the following, we restrict to even dimensions and *even lattices.*

Then, the modular properties of the theta series of such lattices yield constraints for the dimension and the density analogous to Hecke's results for $\ell = 1$ ([C-S1], chapter 7). Still, for some aspects of these questions, the unimodular case remains somewhat special. For example, given a prime ℓ, there exists even ℓ-modular lattices of dimension n if and only if $\ell \equiv 3 \mod 4$ or $n \equiv 0 \mod 4$ (see [Q1]).

4.2 Connection with modular forms. Let Λ be an even lattice of minimum m, and let Θ_Λ be its theta series

$$\Theta_\Lambda(z) = \sum_{x \in \Lambda} q^{(x.x)/2} = 1 + 2sq^{m/2} + \cdots \qquad (\text{ where } q = e^{2\pi i z}).$$

Now, when Λ is ℓ-modular ($\ell > 1$), Θ_Λ must be a modular form of weight $n/2$ with respect to the so-called *Fricke group of level* ℓ, a subgroup of $\mathrm{SL}_2(\mathbb{R})$ which contains $\Gamma_0(\ell)$ with index 2 (here again, the unimodular case is exceptional).

From the algebraic structure of the corresponding space $\mathcal{M}$ of modular forms, Quebbemann derives the notion of extremal modular lattices extending that of [C-S1], chapter 7. Let $d = \dim \mathcal{M}$ be the dimension of $\mathcal{M}$. If a form $f \in \mathcal{M}$ is uniquely determined by the first d coefficients $a_0, a_1, \cdots, a_{d-1}$ of its q-expansion $f = \sum_{k \geq 0} a_k q^k$, the unique form $F_\mathcal{M} = 1 + \sum_{k \geq d} a_k q^k$ is called ℓ. *extremal*, and an even ℓ-modular lattice with this theta series is called an *extremal lattice.* Such a (hypothetical) extremal lattice has the highest possible minimum, equal to $2d$ unless the coefficient a_d of $F_\mathcal{M}$ vanishes. No general results about the coefficients of the extremal modular form and more generally of its eligibility as a theta series seem to be known.

4.3 Special levels. Quebbemann proved that the above method is valid in particular for prime levels ℓ such that $\ell + 1$ divides 24, namely $2, 3, 5, 7, 11$ and 23. (For a more general setup, we refer the reader to [Q1], [Q2] and [S-SP].) The dimension of the space of modular forms is then $d = 1 + \lfloor \frac{n(1+\ell)}{48} \rfloor$ (which reduces to Hecke's result for $\ell = 1$). The proof of the upper bound

$$m \leq 2 + 2 \left\lfloor \frac{n(1+\ell)}{48} \right\rfloor$$

was completed in [S-SP] by R. Scharlau and R. Schulze-Pillot, by investigating the coefficients $a_k,\ k > 0$ of the extremal modular form: all of them are even integers, the leading one a_d is positive, but a_{d+1} is negative for n large enough. So, for a given level in the above list, there are (at most) only finitely many extremal lattices. Other kinds of obstructions may exist.

4.4 Examples.
• $\ell = 7$, at *jump dimensions* (where the minimum may increase) $n \equiv 0 \mod 6$. While a_{d+1} first goes negative at $n = 30$, Scharlau and Hemkemeier proved that no 7-extremal lattice exists in dimension 12: their method consists in classifying for given pairs (n, ℓ) the even lattices Λ of level ℓ (i.e. $\sqrt{\ell}\Lambda^*$ is also even) with $\det \Lambda = \ell^{n/2}$; for $(n, \ell) = (12, 7)$, they found 395 isometry classes, and among them no extremal modular lattice.

If an extremal lattice were to exist for $(n, \ell) = (18, 7)$, it would set new records of density. Bachoc and Venkov proved recently in [B-V2] that no such lattice exists: their proof envolves spherical designs.

• Extremal lattices of jump dimensions are specially wanted, since they often achieve the best known density, like in the following examples:
Minimum 2. $\mathbb{D}_4$ $((n, \ell) = (4, 2))$; $\mathbb{E}_8$ $((n, \ell) = (8, 1))$.
Minimum 4. K_{12} $((n, \ell) = (12, 3))$; BW_{16} $((n, \ell) = (16, 2))$; the Leech lattice $((n, \ell) = (24, 1))$.
Minimum 6. $(n, \ell) = (32, 2)$: 4 known lattices, Quebbemann discovered the first one (denoted Q_{32} in [C-S1]) in 1984; $(n, \ell) = (48, 1)$: 3 known lattices P_{48p}, P_{48q} from coding theory, and a "cyclo-quaternionic" lattice by Nebe.

• Extremal even unimodular lattices are known for any dimension $n \equiv 0 \mod 8$, $n \leq 80$, except for $n = 72$, which would set a new record of density. The case $n = 80$ was recently solved by Bachoc and Nebe. The corresponding Hermite invariant $\gamma = 8$ (largely over the upper bound for period lattices) does not hold the present record for dimension 80, established at $8,0194$ independently by Elkies and Shioda. The same phenomenon appeared at dimension 56.

We give in section 7 Hermitian constructions for most of the above extremal lattices, making obvious their symplectic nature.

5. Voronoi's theory

5.1 Local theory. In section 4, we looked for extremal lattices, which (if any) maximize the Hermite invariant in the (finite) set of modular lattices for a given pair (n, ℓ). In the present section, we go back to the classical notion of an *extreme lattice*, where the Hermite invariant γ achives a local maximum. Here, the existence of such lattices stems from Malher's compactness theorem. The same argument applies when we study the local maxima of density in some natural families of lattices such as isodual lattices, lattices with prescribed automorphisms etc. These families share a common structure: *their connected components are orbits of one lattice under the action of a closed subgroup $\mathcal{G}$ of $\mathrm{GL}(E)$ invariant under transpose.* For such a family $\mathcal{F}$, we can give a unified characterization of the strict local maxima of density. In order to point out the connection with Voronoi's classical theorem

a lattice is extreme if and only if it is perfect and eutactic,

we mostly adopt in the following the point of view of Gram matrices. We denote by $\mathrm{Sym}_n(\mathbb{R})$ the space of $n \times n$ symmetric matrices equipped with the scalar product $< M, N >= \mathrm{Trace}(MN)$. The value at $v \in \mathbb{R}^n$ of a quadratic form A is then ${}^t vAv =< A, v{}^t v >$.

5.2 Perfection, eutaxy and extremality. Let $\mathcal{G}$ be a closed subgroup of $\mathrm{SL}_n(\mathbb{R})$ stable under transpose, and let $\mathcal{F} = \{{}^tPAP, P \in \mathcal{G}\}$ be the orbit of a positive definite matrix $A \in \mathrm{Sym}_n(\mathbb{R})$. We denote by $\mathcal{T}_A$ the *tangent space to the manifold* $\mathcal{F}$ *at* A, and we recall that $S(A)$ stands for the set of the minimal vectors of A.

• Let $v \in \mathbb{R}^n$. The gradient at A (with respect to $<,>$) of the function $\mathcal{F} \to \mathbb{R}^+$ $A \mapsto < A, v{}^tv > \det A^{-1/n}$ is the orthogonal projection $\nabla_v = \mathrm{proj}_{\mathcal{T}_A}(v{}^tv)$ of $v{}^tv$ onto the tangent space at A.

The $\mathcal{F}$-*Voronoi domain* of A is

$$\mathcal{D}_A = \text{ convex hull } \{\nabla_v,\ v \in S(A)\}.$$

We say that A is $\mathcal{F}$-*perfect* if the affine dimension of $\mathcal{D}_A$ is maximum ($= \dim \mathcal{T}_A$), and *eutactic* if the projection of the matrix A^{-1} lies in the interior of $\mathcal{D}_A$.

These definitions reduce to the traditional ones when we take for $\mathcal{F}$ the whole set of positive $n \times n$ matrices (and $\mathcal{T}_A = \mathrm{Sym}_n(\mathbb{R})$). But in this survey we focus on families $\mathcal{F}$ naturally normalized to determinant 1: the tangent space at A to such a family is orthogonal to the line $\mathbb{R}A^{-1}$, and the eutaxy condition reduces to "$0 \in \overset{\circ}{\mathcal{D}}_A$".

• The matrix A is called $\mathcal{F}$-*extreme* if γ achieves a local maximum at A among all matrices in $\mathcal{F}$. We say that A is *strictly* $\mathcal{F}$-*extreme* if this local maximum is strict.

• The above concepts are connected by the following result.

THEOREM ([B-M]). *The matrix A is strictly $\mathcal{F}$-extreme if and only if it is $\mathcal{F}$-perfect and $\mathcal{F}$-eutactic.*

The crucial step in studying the Hermite invariant in an individual family $\mathcal{F}$ is then to check the strictness of any local maximum. A sufficient condition is that any $\mathcal{F}$-extreme matrix should be *well rounded*, i.e. that its minimal vectors should span the space $\mathbb{R}^n$. It was proved by Voronoi in the classical case.

5.3 Isodual lattices. Let σ be an isometry of E with a given integral representation S. Then we can parametrize the family of σ-isodual lattices by the Lie group and symmetrized tangent space at identity

$$\mathcal{G} = \{P \in \mathrm{GL}_n(\mathbb{R}) \mid {}^tP^{-1} = SPS^{-1}\}, \qquad \mathcal{T}_I = \{X \in \mathrm{Sym}_n(\mathbb{R}) \mid SX = -XS\}.$$

The answer to the question

does σ-extremality imply strict σ-extremality?

depends on the representation afforded by $\sigma \in O(E)$. It is positive for symplectic or orthogonal lattices. A minimal counter-example is given by a three-dimensional rotation σ of order 4: the corresponding isodual lattices are decomposable (see [C-S2], th. 1), and the Hermite invariant for this family attains its maximum 1 on a subvariety of dimension 2 (up to isometry).

In [Qi-Z], Voronoi's condition for symplectic lattices was given a suitable complex form. It holds for the Conway and Sloane lattice $M(\mathbb{E}_6)$ (and of course for the Barnes lattice P_6 which is extreme in the classical sense) but not for the lattice $\mathbb{D}_6^+$. (An alternative proof involving differential geometry was given in [Bav1].)

In dimension 5 et 7, the most likely candidates for densest isodual lattices were also discovered by Conway and Sloane; they were successfully tested for isodualities σ of orthogonal type (of respective signatures $(4,1)$ and $(4,3)$). In dimension 3, Conway and Sloane proved by classification and direct calculation that the so called m.c.c. isodual lattice is the densest one (actually, there are only 2 well rounded isodual lattices, m.c.c. and the cubic lattice).

5.4 Extreme modular lattices. The classical theory of extreme lattices was recently revisited by B. Venkov [Ve] in the setting of spherical designs. That the set of minimal vectors of a lattice be a spherical 2- or 4-design is a strong form of the conditions of eutaxy (equal coefficients) or extremality.

An extremal ℓ-modular lattice is not necessarily extreme: the even unimodular lattice $\mathbb{E}_8 \perp \mathbb{E}_8$ has minimum 2, hence is extremal, but as a decomposable lattice, it could not be perfect. By use of the modular properties of some theta series with spherical coefficients, Bachoc and Venkov proved ([B-V2]) that this phenomenon could not appear near the "jump dimensions": in particular, any extremal ℓ-modular lattice of dimension n such that $(\ell = 1, and\ n \equiv 0, 8 \mod 24)$, or $(\ell = 2, and\ n \equiv 0, 4 \mod 16)$, or $(\ell = 3, and\ n \equiv 0, 2 \mod 12)$, is extreme.

This applies to the famous lattices quoted in section 4. [For some of them, alternative proofs of the Voronoi conditions could be done, using the automorphism groups (for eutaxy), testing perfection modulo small primes, or inductively in the case of laminated lattices.]

5.5 Classification of extreme lattices. Voronoi established that there are only finitely many equivalence classes of perfect matrices, and he gave an algorithm for their enumeration.

Let A be a perfect matrix, and $\mathcal{D}_A$ its traditional Voronoi domain. It is a polyhedron of maximal dimension $N = n(n+1)/2$, with a finite number of hyperplane faces. Such a face $\mathcal{H}$ of $\mathcal{D}_A$ is simultaneously a face for the domain of exactly one other perfect matrix, called *the neighbour of A across the face $\mathcal{H}$.*

We get, in taking the dual polyhedron, a graph whose edges describe the neighbouring relations; this graph has finitely many inequivalent vertices. Voronoi proved that this graph is connected, and he used it up to dimension 5 to confirm the classification by Korkine and Zolotarev. His attempt for dimension 6 was completed in 1957 by Barnes. Complete classification for dimension 7 was done by Jaquet in 1991 using this method. Recently implemented by Batut in dimension 8, Voronoi's algorithm produced, by neighbouring only matrices with $s = N, N+1$ and $N+2$, exactly 10916 inequivalent perfect lattices. There may exist some more.

This algorithm was extended in [B-M-S] to matrices invariant under a given finite group $\Gamma \subset \mathrm{GL}_n(\mathbb{Z})$): it works in the centralizer of Γ in $\mathrm{Sym}_n(\mathbb{R})$.

That there are only finitely many isodual-extreme lattices of type symplectic or orthogonal stems from their well roundness. But the present extensions of Voronoi's algorithm are very partial (see section 6).

5.6 Voronoi's paths and isodual lattices. The densest known isodual lattices discovered by Conway and Sloane up to seven dimensions were found on paths connecting, in the lattice space, the densest lattice Λ to its dual Λ^*: such a path turns out to be stable under a fixed duality, and the isodual lattice $M(\Lambda)$ is the fixed point for this involution. In [C-S2], these paths were constructed by gluing theory.

Actually, the Voronoi algorithm for perfect lattices provides another interpretation of them. In dimensions 6 and 7, the densest lattices $\mathbb{E}_6$ and $\mathbb{E}_7$ and their respective duals are Voronoi neighbours of each other. The above mentioned paths $\Lambda - \Lambda^*$ are precisely the corresponding neighbouring paths. For dimensions 3 and 5, we need a group action: for dimension 5, we use the regular representation Γ of the cyclic group of order 5, and the path $\mathbb{D}_5 - \mathbb{D}_5^*$ contains the Γ-neighbouring path leading from $\mathbb{D}_5$ to the perfect lattice $\mathbb{A}_5^3$; for dimension 3, we use the augmentation representation of the cyclic group of order 4, and the Conway and Sloane path $\Lambda - \Lambda^*$ is part of the Γ-neighbouring path leading from $\Lambda = \mathbb{A}_3$ to the Γ-perfect lattice called "axial centered cuboidal" in [C-S2].

5.7 Eutaxy. The first proof of the finiteness of the set of eutactic lattices (for a given dimension and up to similarity) was given by Ash ([A]) by means of Morse theory: the Hermite invariant γ is a topological Morse function, and the eutactic lattices are exactly its non-degenerate critical points. Bavard proved in [Bav1] that γ is no more a Morse function on the space of symplectic lattices of dimension $2g \geq 4$; in particular, one can construct continuous arcs of critical points, such as the following set of symplectic-eutactic 4×4 matrices

$$\{\begin{pmatrix} I & O \\ O & A \end{pmatrix}, \qquad A \in \mathrm{SL}_2(\mathbb{R}) \text{ s.t. } m(A) > 1\}.$$

6. Hyperbolic families of symplectic lattices

This section surveys a recent work by Bavard: in [Bav2] he constructs families of $2g$-dimensional symplectic lattices for which he his able to recover the local and global Voronoi theory, as well as Morse's theory. The convenient frame for these constructions is the Siegel space $\mathfrak{h}_g = \{X + iY \in \mathrm{Sym}_g(\mathbb{C}) \mid Y > 0\}$, modulo homographic action by the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$.

In these families most of the important lattices ($\mathbb{E}_8, K_{12}, BW_{16}$, Leech ...) and many others appear with fine Siegel's representations $Z = X + iY$.

6.1 Definition.

In the following we fix an integral positive symmetric $g \times g$ matrix M.

To any complex number $z = x + iy, y > 0$ in the Poincaré upper half plane $\mathfrak{h}$, we attach the complex matrix $zM = xM + iyM \in \mathfrak{h}_g$, and we consider the family

$$\mathcal{F} = \{zM, z \in \mathfrak{h}\} \subset \mathfrak{h}_g.$$

On can check that the homographic action of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{P}SL_2(\mathbb{R})$ on $\mathfrak{h}$ corresponds to the homographic action of $\begin{pmatrix} \alpha I & \beta M \\ \gamma M^{-1} & \delta I \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathcal{F}$. This last matrix is integral when $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ lies in a convenient congruence subgroup $\Gamma_0(d)$ of $\mathrm{SL}_2(\mathbb{Z})$ (one may take $d = \det M$), thus up to symplectic isometries of lattices, one can restrict the parameter z to a fundamental domain for $\Gamma_0(d)$ in $\mathfrak{h}$.

The symplectic Gram matrix A_z associated to $z = x + iy, y > 0 \in \mathfrak{h}$, as defined in 2.2, is given by

$$A_z = \frac{1}{y}\begin{pmatrix} M^{-1} & xI \\ xI & |z|^2 M \end{pmatrix}.$$

For $g = 1$ and M a positive integer, this is the general 2×2 positive matrix of determinant 1, and we recover the usual representation in $\mathfrak{h}/\mathbb{P}\,\mathrm{SL}_2(\mathbb{Z})$ of the 2-dimensional lattices.

6.2 Voronoi's theory. The Voronoi conditions of eutaxy and perfection for the family $\mathcal{F}$, as defined in 5.2, can be translated in the space $\mathfrak{h}$ of the parameters, equipped with its *Poincaré metric* $ds = \frac{|dz|}{y}$.

• Fix $z \in \mathfrak{h}$, and for any $v \in \mathbb{R}^{2g}$ denote by ∇_v the (hyperbolic) gradient at z of the function $z \mapsto {}^t v A_z v$; we can represent the Voronoi domain of A_z by the convex hull $\mathcal{D}_z$ in $\mathbb{C}$ of the $\nabla_v, v \in S(A_z)$; it has affine dimension $0, 1$ or 2, this maximal value means "perfection" for z. As defined in 5.2, z is eutactic if there exist strictly positive coefficients c_v such that $\sum_{v \in S(A_z)} c_v \nabla_v = 0$.

Bavard showed that Voronoi's and Ash's theories hold for the family $\mathcal{F}$:

Strict extremality ⇔ extremality ⇔ perfection and eutaxy,

Hermite's function is a Morse function, its critical points are the eutactic ones.

Actually, these results are connected to the strict convexity of the Hermite function on the family $\mathcal{F}$ (see [Bav1] for a more general setting).

Remark. In the above theory, the only gradients that matter are the extremal points of the convex $\mathcal{D}_z$. Following Bavard, we call *principal* the corresponding minimal vectors. In the classical theory, all minimal vectors are principal; this is no more true in its various extensions.

• The next step towards a global study of γ in $\mathcal{F}$ was to get an hyperbolic interpretation of the values ${}^t v A_z v$. To this purpose, Bavard represents any vector $v \in \mathbb{R}^{2g}$ by a point $p \in \mathfrak{h} \cup \partial\mathfrak{h}$ in such a way that for all $z \in \mathfrak{h}$, ${}^t v A_z v$ is an exponential function of the hyperbolic distance $d(p, z)$ (suitably extended to the boundary $\partial\mathfrak{h}$). In particular, there is a discrete set $\mathcal{P}$ corresponding to the principal minimal vectors.

• There is now a simple description of the Voronoi theory for the family $\mathcal{F}$. We consider the Dirichlet-Voronoi tiling of the metric space $(\mathfrak{h}, d)$ attached to the set $\mathcal{P}$: the cell around p is $C_p = \{z \in \mathfrak{h} \mid \ d(z,p) \leq d(z,q) \text{ for all } q \in \mathcal{P}\}$.

We then introduce the dual partition: the Delaunay cell of $z \in \mathfrak{h}$ is the convex hull of the points of $\mathcal{P}$ closest to z (for the Poincaré metric), hence it can figure the Voronoi domain $\mathcal{D}_z$. As one can imagine, the $\mathcal{F}$-perfect points are the vertices of the Dirichlet-Voronoi tiling, and the $\mathcal{F}$-eutactic points are those which lie in the interior of their Delaunay cell.

The 1-skeleton of the Dirichlet-Voronoi tiling is the graph of the neighbouring relation between perfect points. Bavard proved that it is connected, and finite modulo the convenient congruence subgroup. For a detailed description of the algorithm, we refer the reader to [Bav2], 1.5 and 1.6.

6.3 Some examples.

• For $M = \mathbb{D}_g$ $(g \geq 3)$, the algorithm only produces one perfect point $z = \frac{1+i}{2}$, corresponding to the so-called lattice $\mathbb{D}_{2g}^{+}$ (see next section).

• The choice $M = \mathbb{A}_g, g \geq 1$ is much less disappointing: it produces many symplectic-extreme lattices, among them $\mathbb{A}_2, \mathbb{D}_4, P_6, \mathbb{E}_8, K_{12}$.

• The densest lattices in the families attached to the Barnes lattices $M = P_g, g = 8, 12$ are the Barnes-Wall lattice BW_{16} and the Leech lattice.

• However, the union of the hyperbolic families of given dimension $2g$ has only dimension $g(g+1)/2 \ +1$; hence it is no wonder that it misses some beautiful symplectic lattices, for instance the lattice $M(\mathbb{E}_6)$.

7. Other constructions

7.1 Hermitian lattices. Let K be a C.M. field or a totally definite quaternion algebra, and let $\mathfrak{M}$ be a maximal order of K. All the above-mentioned famous lattices (in even dimensions) can be constructed as $\mathfrak{M}$-modules of rank k equipped with the scalar product $\mathrm{trace}(\alpha x.\overline{y})$, where trace is the reduced trace $K/\mathbb{Q}$, $x.\overline{y}$ the standard Hermitian inner product on $\mathbb{R}\otimes_{\mathbb{Q}} K^k$ and $\alpha \in K$ some convenient totally positive element (see for example [Bay]). We see in the following examples that such a construction often provides natural symplectic isodualities and automorphisms.

Lattices $\mathbb{D}_{2g}^{+}, g \geq 3$. Here $\mathfrak{M} = \mathbb{Z}[i] \subset \mathbb{C}$ is the ring of Gaussian integers. We consider in $\mathbb{C}^g$ equipped with the scalar product $\frac{1}{2}\,\mathrm{trace}(x.\overline{y})$ the lattice $\{x=(x_1,x_2,\cdots,x_g)\in\mathfrak{M}^g \mid x_1+x_2+\cdots+x_g\equiv 0 \mod (1+i)\}$ which is isometric to the root lattice $\mathbb{D}_{2g}$. Now we consider the conjugate elements $e=\frac{1}{1+i}(1,1,\cdots,1)$ and $\overline{e}$ $(= ie = (1,1,...,1)-e)$ of $\mathbb{C}^g$; then the sets $\mathbb{D}_{2g}^{+} = \mathbb{D}_{2g}\cup(e+\mathbb{D}_{2g})$ and $\mathbb{D}_{2g}^{-} = \mathbb{D}_{2g}\cup(\overline{e}+\mathbb{D}_{2g})$ turn out to be dual lattices, that coincide when g is even. In any case, the multiplication by i provides a symplectic isoduality. An obvious group of Hermitian automorphisms consist of permutations of the x_i's and even sign changes. Thus, comparing its order $2^g g!$ to the Hurwitz bound $(2)84(g-1)$, one sees that, except for $g=3$, no lattice of the family is a Jacobian (in the opposite direction, all lattices, except for $g=3$, are extreme in the Voronoi sense).

Barnes-Wall lattices $BW_{2^k}, k\geq 2$. Here, K is the quaternion field $\mathbb{Q}_{2,\infty}$ defined over $\mathbb{Q}$ by elements i,j such that $i^2=j^2=-1, ji=-ij$, $\mathfrak{M}$ is the Hurwitz order ($\mathbb{Z}$-module generated by $(1,i,j,\omega)$ where $\omega = 1/2(1+i+j+ij)$), and we consider the two-sided ideal $\mathfrak{A}=(1+i)\mathfrak{M}$ of $\mathfrak{M}$. Starting from $M_0=\mathfrak{A}$, we define inductively the right and left $\mathfrak{M}$-modules

$$M_{k+1} = \{(x,y)\in M_k\times M_k \mid x\equiv y \mod \mathfrak{A}M_k\} \quad \subset K^{2^{k+2}},$$

and for k odd (resp. even) we put $L_k = M_k$ (resp. $\mathfrak{A}^{-1}M_k$). For the scalar product $\frac{1}{2}\,\mathrm{trace}(x.\overline{y})$, we have $L_0\sim\mathbb{D}_4$, $L_1\sim\mathbb{E}_8$ and generally $L_k\sim BW_{2^{2k+2}}$. These lattices are alternatively 2-modular and unimodular: the right multiplication by i (resp. $j-i$) for k odd (resp. k even) provides a symplectic similarity σ from $L_k^* = L_k$ (resp. $\mathfrak{A}^{-1}L_k$) onto L_k. Using their logarithmic bound for the density of a period lattice, Buser and Sarnak proved that the Barnes-Wall lattices are certainely not Jacobians for $k\geq 5$. As usual, an argument of automorphisms extends this result for $1\leq k\leq 4$: the group $\mathrm{Aut}_\sigma(L_k)$ embeds diagonally into $\mathrm{Aut}_\sigma(L_{k+2})$, and adding transpositions and sign changes, one obtains a subgroup of $\mathrm{Aut}_\sigma(L_{k+2})$ of order $2^7\times\mathrm{Aut}_\sigma(L_k)$; starting from the subgroup of automorphisms of $\mathbb{D}_4$ given by left multiplication by the 24 units of $\mathfrak{M}$, or from the group $\mathrm{Aut}_\sigma(\mathbb{E}_8)$, one sees that $|\,\mathrm{Aut}_\sigma(L_k)|$ is largely over the Hurwitz bound.

Hermitian extensions of scalars. Let $\mathfrak{M}$ the ring of integers of a imaginary quadratic field. Following [G], Bachoc and Nebe show in [B-N] that by tensoring over $\mathfrak{M}$ a modular $\mathfrak{M}$-lattice, one can shift from one level to another; this construction preserves the symplectic nature of the isoduality, and hopefully, the minimum. For this last question, we refer to the preprint [Cou].

In [B-N], $\mathfrak{M}$ is the ring of integers of the quadratic field of discriminant -7. Let L_r be an $\mathfrak{M}$-lattice of rank r, unimodular with respect to its Hermitian structure, and consider the $\mathfrak{M}$-lattices $L_{2r} = (\mathbb{A}_2 \perp \mathbb{A}_2) \otimes_{\mathfrak{M}} L_r$ and $L_{4r} = \mathbb{E}_8 \otimes_{\mathfrak{M}} L_r$. By a determinant argument, one sees that (for the usual scalar product) the $\mathbb{Z}$-lattices L_r, L_{2r} and L_{4r} are symplectic modular lattices of respective levels $7, 3$ and 1. Starting from the Barnes lattice P_6, Gross obtained the Coxeter-Todd lattice K_{12} and the Leech lattice, of minimum 4. The same procedure was applied in [B-N] to a 20-dimensional lattice appearing in the ATLAS in connection with the Mathieu goup M_{22}, and led to the first known extremal modular lattices of minimum 8, and respective dimensions 40 and 80. Note that while coding theory was involved in the original proof of the extremality of the unimodular lattices of dimension 80, an alternative "à la Kitaoka" proof is given in [Cou].

7.2 Exterior power. Let $1 \leq k < n$ be two integers, and let E be a Euclidean space of dimension n; its exterior powers carry a natural scalar product which makes the canonical map $\sigma: \bigwedge^{n-k} E \to (\bigwedge^k E)^*$ an isometry $\bigwedge^{n-k} E \to (\bigwedge^k E)$ of square $(-1)^{k(n-k)}$. Let L be a lattice in E. It is shown in [Cou1] that σ maps the lattice $\bigwedge^{n-k} L$ onto $\sqrt{\det L}(\bigwedge^k L)^*$. In particular, when $n = 2k$, σ is a symplectic or orthogonal similarity of the $\binom{2k}{k}$-dimensional lattice $(\bigwedge^k L)$ onto its dual; if moreover L is integral, the lattice $(\bigwedge^k L)$ is modular of level $\det L$.

For instance, the lattice $\bigwedge^2 \mathbb{D}_4$ is isometric to $\mathbb{D}_6^+$, and the lattice $\bigwedge^3 \mathbb{E}_6$, in 20 dimensions, is 3-modular of symplectic type, with minimum 4, thus extremal.

Remark. Exterior even powers of unimodular lattices are unimodular lattices of special interest for the theory of group representations. Let us come back to the notation of this subsection. For even k, the canonical map $\operatorname{Aut} L \to \operatorname{Aut} \bigwedge^k L$ has kernel ± 1, and induces an embedding $\operatorname{Aut} L/(\pm 1) \hookrightarrow \operatorname{Aut} \bigwedge^k L$. Actually, the exterior squares of the lattice $\mathbb{E}_8$ and the Leech lattice provide faithful representations of minimal degrees of the group $O_8^+(2)$ and of the Conway group Co_1 respectively.

7.3 Group representations. Many important symplectic modular lattices were discovered by Nebe, Plesken ([N-P]) and Souvignier ([Sou]) while investigating finite rational matrix groups. In [S-T], Scharlau and Tiep, using symplectic groups over $\mathbb{F}_p$, construct large families of symplectic unimodular lattices, among them that of dimension 28 discovered by combinatorial devices by Bacher and Venkov in [B-V1].

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