ON WEAKLY EUTACTIC FORMS

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Abstract. We make precise some properties of the Hermite function in relation with Morse theory introduced by Avner Ash in his papers [Ash1] and [Ash2] and with the cellular decomposition of the space of positive definite quadratic forms. We also establish a link between Ash's and Bavard's mass formulae.

1. Introduction

The space $\mathrm{Sym_n}(\mathbb{R})$ of $n \times n$ real symmetric matrices is equipped with the scalar product

$$\langle A, B\rangle = \mathrm{Tr}(AB).$$

With any column vector $x \in \mathbb{R}^n$ we associate the matrix $x\,{}^t\!x \in \mathrm{Sym_n}(\mathbb{R})$, and for a symmetric subset S of $\mathbb{R}^n$, the *perfection rank* $\mathrm{perf\,rk}\, S$ of S is the dimension, often denoted by r, of the span V_S of $x\,{}^t\!x$, $x \in S/\pm$.

Let $C \subset \mathrm{Sym_n}(\mathbb{R})$ be the open cone of positive definite matrices. With a matrix $A \in C$ we associate the quadratic form Q

$$x \in \mathbb{R}^n \mapsto {}^t\!xAx = \langle A, x\,{}^t\!x\rangle\,,$$

its *minimum* $\min A$ on non-zero integral vectors, the set

$$S = S(A) = \{x \in \mathbb{Z}^n \mid \langle A, x\,{}^t\!x\rangle = \min A\}$$

of its *minimal vectors*, and the (half) *kissing number* $s = \frac{1}{2}\,|S|$. We shall often denote by V_A the space $V_{S(A)}$. Set

$$K = \{M \in C \mid \min M \geq 1\}.$$

Let $A \in \partial K$ (the boundary of K) be a matrix with minimum 1 and set of minimal vectors $S = S(A)$. The *cell* $\mathcal{C}_A$ of A is the following subset of ∂K:

$$\mathcal{C}_A = \{M \in C \mid \min M = 1 \text{ and } S(M) = S\}\,,$$

with closure

$$\overline{\mathcal{C}_A} = \{M \in C \mid \min M = 1 \text{ and } S(M) \supset S\}\,.$$

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We say that the cell $\mathcal{C}_{A'}$ is *below* the cell $\mathcal{C}_A$ (notation: $\mathcal{C}_{A'} \prec \mathcal{C}_A$) if $S(A') \subset S(A)$. The cell $\mathcal{C}_A$ is open in its closure $\overline{\mathcal{C}_A} = \cup_{\mathcal{C}_A \prec \mathcal{C}_{A'}} \mathcal{C}_{A'}$. The cells and the faces of K are intimately related: the faces of K are the closures of the cells; see Proposition 2.2.

With $A \in \partial K$, the also associate its *Voronoi domain* $\mathcal{D}_A$, the convex hull of the half-lines $\mathbb{R}_{\geq 0}\, x^t x$, $x \in S(A)$. All matrices in a given cell $\mathcal{C}$ have the same Voronoi domain, that we denote by $\mathcal{D}_{\mathcal{C}}$. We shall see that given $A \in \partial K$, the map $\mathcal{C} \mapsto \mathcal{D}_{\mathcal{C}}$ is a one-to-one correspondence between cells $\mathcal{C} \prec \mathcal{C}_A$ and the faces of the polyhedral cone $\mathcal{D}_A$; see Lemma 2.1. Note that M belongs to the affine span of $\mathcal{C}_A$ if and only if $\langle M - A, x^t x\rangle$ is zero on $x^t x$, $x \in S(A)$, i.e. if and only if $M - A$ is orthogonal to $\mathcal{D}_A$. Thus, the perfection rank of A is

$$\dim \mathcal{D}_A = \frac{n(n+1)}{2} - \dim \mathcal{C}_A\,.$$

Eventually, recall that A is called *weakly eutactic* (resp. *semi-eutactic*, resp. *eutactic*) if there are real numbers (resp. non-negative numbers, resp. strictly positive numbers) a_x, $x \in S$, such that

$$A^{-1} = \sum_{x \in S/\pm} a_x x^t x\,.$$

In practice, we shall essentially consider *well-rounded matrices* (and *cells*), those whose minimal vectors span $\mathbb{R}^n$. Weakly eutactic matrices are well rounded; see Proposition 3.8 for a more precise statement.

Let

$$\gamma(A) = \frac{\min A}{(\det A)^{1/n}}$$

be the Hermite function. In [Ash1], Ash proves that the "packing function" $P(A) = \gamma(A)^{-n}$ is a topological Morse function whose non-degenerate critical points are exactly the eutactic matrices A. In [B-M1] we proved that each cell contains at most one weakly eutactic matrix which (if any) is exactly the matrix at which the strictly convex function γ (or $\frac{1}{\det}$) attains its minimum on the cell. Our first aim is to specify which kind of topological ordinary points are the weakly eutactic, non-eutactic matrices. We make use of the notions of semi- and weak eutaxy, which were introduced after Ash's paper was written (in [C-S], 1988, and [B-M1], 1996, respectively).

Theorem 1.1. *Let $A \in \partial K$ be a matrix of minimum* 1*, with set of minimal vectors S. Let*

$$H_A = \{M \in \mathrm{Sym_n}(\mathbb{R}) : \langle M - A, A^{-1}\rangle = 0\}$$

be the tangent hyperplane to the determinant surface at A, and define the open half-spaces

$$H_A^+ = \{M \in \mathrm{Sym_n}(\mathbb{R}) \mid \langle M - A, A^{-1} \rangle > 0\}\,,$$
$$H_A^- = \{M \in \mathrm{Sym_n}(\mathbb{R}) \mid \langle M - A, A^{-1} \rangle < 0\}\,.$$

Then:

1. *A is weakly eutactic if and only if its cell $\mathcal{C}_A$ is contained in H_A.*
2. *A is semi-eutactic if and only if H_A supports K (i.e. $K \subset \overline{H_A^+}$).*
3. *A is eutactic if and only if H_A supports K and $K \cap H_A = \overline{\mathcal{C}}_A$.*

More precisely, if A is weakly eutactic but not semi-eutactic, H_A^- contains a cell below that of A, and if A is semi-eutactic but not eutactic, H_A contains a cell strictly below that of A, and indeed a unique maximal one, namely the relative interior of $K \cap H_A$.

[Note that an example of a matrix A with H_A supporting K but $K \cap H_A \supsetneqq \overline{\mathcal{C}_A}$ was given by Ash in [Ash1] (it is the unique semi-eutactic, non-eutactic matrix for $n \leq 4$).]

Our second aim is to study the connection between the two known "mass formulae with signs" which occur in our context (Ash, [Ash2]; Bavard, [Bv]). Both formulae take the form (for $n \geq 2$)

$$\sum_{\mathcal{C}/\sim} \frac{(-1)^{\dim \mathcal{C}}}{|\mathrm{Aut}^+(\mathcal{C})|} = \chi(\mathrm{SL}_n(\mathbb{Z}))\,,$$

but have distinct ranges of summation: all well-rounded cells up to equivalence in Bavard's formula, only those which contain a eutactic form in Ash's formula; see Section 3. (The right hand side is equal to $-\frac{1}{12}$ for $n = 2$ and to 0 for $n \geq 3$.) We remark that on a well-rounded cell without a weakly eutactic matrix, $\inf_{\mathcal{C}} \gamma$ is attained at a unique matrix $A \in \partial\mathcal{C}$. We shall prove that this matrix is weakly eutactic, but not eutactic.

Consequently, the difference of the left-hand sides of the formulae above may be written in the form

$$\sum_{A} \sum_{\mathcal{C}/\sim} \frac{(-1)^{\dim \mathcal{C}}}{|\mathrm{Aut}^+(\mathcal{C})|} = 0\,,$$

where A runs through the set of (equivalence classes of) weakly eutactic, non-eutactic matrices, and $\mathcal{C}$ through the set $\mathcal{E}_A$ of cells such that $\inf_{\mathcal{C}} \gamma$ is attained on A. This strongly suggests the following theorem:

Theorem 1.2. *For every weakly eutactic, non-eutactic matrix A,*

$$\sum_{\mathcal{C}\in\mathcal{E}_A/\sim} \frac{(-1)^{\dim\mathcal{C}}}{|\mathrm{Aut}^+(\mathcal{C})|} = 0\,.$$

This theorem results in dimensions $n \le 5$ from Batut's paper [Bt] where he classifies cells and weakly eutactic matrices.

Theorem 1.1 is proved in Section 2, together with some complements. In Section 3, we reduce Theorem 1.2 to a purely combinatorial statement, that we prove by applying a generalized Euler formula for polytopes to an extension $\mathcal{D}'_A$ of the Voronoi domain $\mathcal{D}_A$. Finally, a short fourth section is devoted to algorithmic remarks.

2. Weak eutaxy and Morse theory

We still consider a positive definite matrix A, that we scale to minimum 1 (i.e., $A \in \partial K$), and denote by $S = S(A)$ its set of minimal vectors. Recall that the *Voronoi domain of* A is the convex hull $\mathcal{D}_A$ of the half-lines $\mathbb{R}_{\ge 0}\, x\,{}^t\!x$, $x \in S$. Thus A is weakly eutactic if A^{-1} lies in V_A, the real span of $\mathcal{D}_A$, and semi-eutactic (resp. eutactic) if A^{-1} lies in $\mathcal{D}_A$ (resp. in the relative interior of $\mathcal{D}_A$).

Lemma 2.1 below shows that there is a one-to-one correspondence between the set of non-zero faces of $\mathcal{D}_A$ and the set of cells $\mathcal{C} \prec \mathcal{C}_A$.

Lemma 2.1. *Let $A \in \partial K$ with Voronoi domain $\mathcal{D}_A$, and let $\mathcal{C} \prec \mathcal{C}_A$. Then its Voronoi domain $\mathcal{D}_\mathcal{C}$ is a face of $\mathcal{D}_A$; more precisely, if $\mathcal{C} \ne \mathcal{C}_A$, for every $M \in \mathcal{C}$, the hyperplane $\mathcal{H} = (M - A)^\perp$ supports $\mathcal{D}_A$ and $\mathcal{H} \cap \mathcal{D}_A = \mathcal{D}_\mathcal{C}$. Conversely, any face $\mathcal{F}$ of $\mathcal{D}_A$ is the Voronoi domain $\mathcal{D}_\mathcal{C}$ of the cell $\mathcal{C} \prec \mathcal{C}_A$ of any matrix $M = A + \varepsilon F$ where F is orthogonal to a supporting hyperplane of $\mathcal{C}_A$ for $\mathcal{F}$ and $\varepsilon > 0$ is small enough.* [In the correspondence above, the improper face $\mathcal{D}_A$ corresponds to $\mathcal{C}_A$.]

Proof. Let $\mathcal{C} \precneqq \mathcal{C}_A$, $M \in \mathcal{C}$, $F = M - A$ and $\mathcal{H} = F^\perp$. For $x \in S(A)$, we have $\langle F, x\,{}^t\!x\rangle = \langle M, x\,{}^t\!x\rangle - 1 \ge 0$ and equality holds if and only if $x \in S(M)$. For $v = \sum_{x\in S(A)/\pm} a_x\, x\,{}^t\!x \in \mathcal{D}_A$ ($a_x \ge 0$),

$$\langle v, F\rangle = \sum_{x\in S(A)\smallsetminus S(M)} a_x\, \langle x\,{}^t\!x, F\rangle$$

vanishes if and only if the a_x are zero on $S(A)\smallsetminus S(M)$, i.e. if and only if $v \in \mathcal{D}_M$. This shows that $\mathcal{H}$ supports $\mathcal{D}_A$ with corresponding face $\mathcal{D}_M = \mathcal{D}_\mathcal{C}$. [Replacing $M - A$ by its orthogonal projection to V_A, we could have assumed that $F \in V_A$.]

Conversely, let $\mathcal{F}$ be a proper face of $\mathcal{D}_A$ with supporting hyperplane $\mathcal{H} = F^\perp$ for some F with $\langle F, \mathcal{D}_A\rangle \geq 0$. Let $M = A + \varepsilon F$ where $\varepsilon > 0$ is small enough to ensure that $S(M) \subset S(A)$. For $x \in S(A)$, we have $\langle M, x^t x\rangle = 1 + \varepsilon\langle F, x^t x\rangle \geq 1$, with equality if and only if $x^t x \in \mathcal{H} \cap \mathcal{D}_A = \mathcal{F}$. Hence $\mathcal{F}$ is the Voronoi domain of the cell of M. □

Proposition 2.2. *The closures of the cells are the faces of K. In particular, the smallest face of K containing a given matrix $A \in \partial K$ is the closure $\overline{\mathcal{C}_A}$ of its cell.*

Proof. (1) Let $\mathcal{C} = \mathcal{C}_A$ the cell of a matrix $A \in \partial K$ (thus $\min A = 1$), and let $S = S(A)$. Set $F = \frac{1}{s}\sum_{x\in S/\pm} x^t x$ and let $H = A + F^\perp$ the affine hyperplane through A orthogonal to F. For all $M \in K$, we have

$$\langle M - A, F\rangle = \frac{1}{s}\sum_{x\in S/\pm}(\langle M, x^t x\rangle - 1) \geq 0 \qquad (*)$$

(because $M \in K$), which shows that $K \subset \overline{H^+}$, i.e. that H supports K. Moreover, $(*)$ shows that M belongs to H if and only if $\langle M, x^t x\rangle = 1$ for all $x \in S$, i.e. $\min M = 1$ and $S(M) \supset S$, or otherwise stated, that M belongs to $\overline{\mathcal{C}}$. Finally, $\overline{\mathcal{C}} = K \cap H$ is a face of K.

(2) Let $\mathcal{F}$ be a face of K, H a supporting hyperplane of K such that $\mathcal{F} = K \cap H$, and $F \in H^\perp$ pointing towards K. Let $A \in \mathcal{F}$, with Voronoi domain $\mathcal{D}_A$ and set of minimal vectors S. We first prove that F belongs to $\mathcal{D}_A$.

Consider the cone Γ defined by the conditions $\langle -F, v\rangle \geq 0$ and $\forall\, x \in S$, $\langle x^t x, v\rangle \geq 0$. Let $v \in \Gamma$. For $\varepsilon > 0$ small enough, $M = A + \varepsilon v$ belongs to K (we have $S(M) \subset S(A)$ because ε is small and $\langle M, x^t x\rangle = 1 + \varepsilon\langle v, x^t x\rangle \geq 1$ for all $x \in S$). Since H supports K, we have $\langle F, v\rangle \geq 0$, hence Γ is included in H. Since the dimension of Γ is not maximal, Voronoi's *principe fondamental* ([V], § 8, p. 113) shows that the vectors $-F, x^t x$, $x \in S$ satisfy a relation of the form $\rho(-F) + \sum \rho_x\, x^t x = 0$ with non-negative, not all zero coefficients ρ, ρ_x. If ρ were zero, we would have $\sum \rho_x = \langle A, \sum \rho_x x^t x\rangle = 0$ and ρ and the ρ_x would be zero. Finally, we obtain a relation

$$F = \sum_{x\in S} \alpha_x\, x^t x$$

with non-negative α_x, which shows that $F \in \mathcal{D}_A$.

Let $\mathcal{F}'$ be the smallest face of $\mathcal{D}_A$ containing F (maybe, $\mathcal{D}_A$ itself), and let $S' = \{x \in S \mid x^t x \in \mathcal{F}'\}$. Actually F belongs to the relative interior of $\mathcal{F}'$, and hence there is a relation $F = \sum_{x\in S'} \lambda_x\, x^t x$ with *strictly* positive coefficients λ_x. Let M be a matrix in K. Then we have

$\langle M-A,F\rangle = \sum_{x\in S'} \lambda_x \langle M-A, x^t\!x\rangle$ with all terms $\langle M-A, x^t\!x\rangle \geq 0$ (because $M \in K$), so that

$$H \cap K = \{M \in K \mid \forall\, x \in S', \langle M-A, x^t\!x\rangle = 0\} = \{M \in K \mid S(M) \supset S'\}$$

is the closure of the cell $\mathcal{C}' \prec \mathcal{C}_A$ associated with the face $\mathcal{F}'$. □

Remark 2.3. It results from the proof above that, with its notation, we have

$$A \in \mathrm{Int}(H \cap K) \iff H \cap K = \overline{\mathcal{C}_A} \iff F \in \mathrm{Int}(\mathcal{D}_A)\,.$$

[The notation Int stands for the relative interior.]

Proof of Theorem 1.1.

1. We now locate with respect to H_A the cell $\mathcal{C}_A$ of A or equivalently its affine span

$$\mathrm{Aff}(\mathcal{C}_A) = \{M \mid \forall x \in S, \langle M-A, x^t\!x\rangle = 0\} = A + V_A^{\perp}\,.$$

(V_A is the span of $\mathcal{D}_A$.) Then A is weakly eutactic if and only if A^{-1} belongs to V_A, i.e. (by taking the orthogonal complements) if and only if $H_A \supset \mathcal{C}_A$, which proves the first statement of Theorem 1.1.

From now on, we assume that A is weakly eutactic.

2. (a) If A is semi-eutactic, write $A^{-1} = \sum a_x\, x^t\!x$ with non-negative a_x. Let $M \in K$. For every $x \in S(A)$, we have $\langle M, x^t\!x\rangle \geq 1$, $\langle A, x^t\!x\rangle = 1$, hence $\langle M-A, x^t\!x\rangle \geq 0$, whence $\langle M-A, A^{-1}\rangle \geq 0$, i.e. M lies in $\overline{H_A^+}$. This shows that H_A supports K.

(b) If A is not semi-eutactic, since the closed convex cone $\mathcal{D}_A$ is the intersection of its supporting half-spaces (see [Br], I, Th. 4.5), there exists a facet whose hyperplane separates A^{-1} and $\mathcal{D}_A$, i.e. a vector F such that $\langle F, A^{-1}\rangle < 0$ and $\langle F, x^t\!x\rangle \geq 0$ for all $x \in S(A)$. The cell corresponding to the facet above by Lemma 2.1 is contained in H_A^-, and in particular, H_A does not support K.

From now on, we assume that A is semi-eutactic, and hence that H_A supports K.

3. (a) This time, we consider a eutaxy relation with strictly positive coefficients a_x. As in 2. (a), for any $M \in K$, we have $\langle M-A, A^{-1}\rangle = \sum_{x\in S} a_x \langle M-A, x^t\!x\rangle$ with all $\langle M-A, x^t\!x\rangle \geq 0$. Hence,

$$M \in H_A \iff \forall\, x \in S,\ \langle M-A, x^t\!x\rangle = 0 \iff S(M) \supset S\,,$$

which shows that $K \cap H_A = \overline{\mathcal{C}_A}$.

(b) If A is not eutactic, let $\mathcal{F}$ be a proper face of $\mathcal{D}_A$ containing A^{-1}. There exists a eutaxy relation $A^{-1} = \sum_{x^t x \in \mathcal{F}} a_x\, x^t\!x$ with non-negative a_x. The cell $\mathcal{C} \prec_{\neq} \mathcal{C}_A$ associated with $\mathcal{F}$ is contained in H_A: let $M \in \mathcal{C}$; $\langle M-A, x^t\!x\rangle$ is zero if $x^t\!x \in \mathcal{F}$, and then we have $\langle M-A, A^{-1}\rangle = 0$. □

We end this section with some more precisions on semi-eutaxy.

Proposition 2.4. *Suppose that A is semi-eutactic and let $\mathcal{F}$ be the smallest face of $\mathcal{D}_A$ containing A^{-1} ($\mathcal{F} = \mathcal{D}_A$ if A is eutactic). Let*

$$T = \{x \in S(A) \mid x^t x \notin \mathcal{F}\}\,.$$

Then for every eutaxy relation $A^{-1} = \sum_{x \in S} a_x\, x^t x$ with all $a_x \geq 0$, we have $a_x = 0$ on T, and there exists such a relation whose coefficients are strictly positive on $S(A) \smallsetminus T$.

Proof. Let H be a supporting hyperplane of $\mathcal{F}$ and $F \in H^\perp$ pointing to $\mathcal{D}_A$. Write $A^{-1} = B + C$ with

$$B = \sum_{x \in S \smallsetminus T} a_x\, x^t x \quad \text{and} \quad C = \sum_{x \in T} a_x\, x^t x\,.$$

Then $\langle A^{-1}, F\rangle = \langle B, F\rangle = 0$, hence $\langle C, F\rangle = \sum_{x \in T} a_x\, \langle x^t x, F\rangle = 0$. Since the $\langle x^t x, F\rangle$ are strictly positive on T, all a_x are zero on T.

Since the face $\mathcal{F}$ is minimal among those which contain A^{-1}, A^{-1} lies in the relative interior of $\mathcal{F}$, which shows that it can be written as a linear combination of the $x^t x$, $x \in S \smallsetminus T$ with strictly positive coefficients. □

3. Ash's and Bavard's mass formulae with signs.

In this section, all matrices we consider are assumed to be scaled to minimum 1, so that the determinant and Hermite functions are connected by the relation $\det(M) = \gamma(M)^{-n}$.

Note that in Ash's original formula

$$\sum_{A \text{ eut. } /\sim} \frac{(-1)^{i(A)}}{|\mathrm{Aut}^+(A)|} = \chi(\mathrm{SL}_n(\mathbb{Z}))\,,$$

the index $i(A)$ of the eutactic matrix A is the dimension of the smallest face of K containing A, i.e. $\dim \mathcal{C}_A$, as stated in the introduction; see Proposition 2.2.

Lemma 3.1. *A cell $\mathcal{C}$ is well rounded if and only if the determinant is bounded above on $\mathcal{C}$.*

Proof. Hadamard's inequality (see [M], Theorem 2.2.1) shows that the determinant is bounded above on any well-rounded cell. Conversely, suppose that $\mathcal{C}$ is not well rounded. Up to integral equivalence, we may assume that the last coordinate of every $x \in S$ is zero. Let $M \in \mathcal{C}$, and for $\lambda \geq 1$, denote by Q_λ the diagonal matrix with diagonal $(1, \dots, 1, \lambda)$. Then $M_\lambda = {}^t Q_\lambda M Q_\lambda$ belongs to $\mathcal{C}$ and we have

$$\det M_\lambda = \lambda^2 \det M \underset{\lambda \to \infty}{\longrightarrow} \infty\,. \quad \square$$

We know that $\mathcal{C}$ is well rounded if and only if $\overline{\mathcal{C}}$ is compact, that the Hermite invariant γ has a minimum on $\overline{\mathcal{C}}$ if and only if $\mathcal{C}$ is well rounded (by the lemma above), and that this minimum is then attained at a unique matrix $M \in \overline{\mathcal{C}}$, which is the unique weakly eutactic matrix in its cell ([B-M1], Th. 3.4 and 3.5).

Theorem 1.1 shows that given a weakly eutactic, non-eutactic matrix A there exist cells $\mathcal{C} \precneqq \mathcal{C}_A$ contained in the closed half-space $\overline{H_A^-}$. (The notation $\mathcal{C}_A, H_A, H_A^-, \dots$ is that of the previous section.) For such cells, we prove (compare [Bt]):

Theorem 3.2. *With the notation above, suppose A is weakly eutactic non-eutactic, and let $\mathcal{C} \precneqq \mathcal{C}_A$ be a cell below the cell of A. Then the (unique) minimum on $\overline{\mathcal{C}}$ of the Hermite function is attained at A if and only if $\mathcal{C}$ is included in the closed half-space $\overline{H_A^-}$, and then $\mathcal{C}$ is well rounded and contains no weakly eutactic matrix.*

Conversely, for every well rounded cell $\mathcal{C}$ without a weakly eutactic matrix, the minimum on $\overline{\mathcal{C}}$ of γ is attained at a weakly eutactic, non-eutactic *matrix A.*

Otherwise stated, there is a partition of the set of well-rounded cells without a eutactic matrix, each part of which contains a unique cell having a weakly eutactic matrix. (The parts are the sets $\mathcal{E}_A$ for A weakly eutactic, non-eutactic.)

Proof. Let A be a weakly eutactic, non-eutactic matrix. Recall the well known formula in which F stands for any symmetric matrix:

$$\det(A+tF) = \det A\,(1+t\langle F, A^{-1}\rangle+\frac{1}{2}\,(\langle F, A^{-1}\rangle^2-\langle FA^{-1}, FA^{-1}\rangle)\,t^2+O(t^3))\,.$$

(Formulae of this kind occur e.g. in [Ash1], Corollary to Proposition 3.) Since A is positive definite and F is symmetric, the eigenvalues of FA^{-1} are real. Hence for small enough $t > 0$, we have $\det(A + tF) > \det(A)$ if $A + tF \in H_A^+$ and $\det(A + tF) < \det A$ if $A + tF \in \overline{H_A^-}$.

Let $\mathcal{C} \precneqq \mathcal{C}_A$. Since $A \in \overline{\mathcal{C}} \cap \overline{H_A^+}$ the first inequality above shows that if the determinant is smaller than $\det A$ on $\mathcal{C}$, then $\mathcal{C}$ is included in $\overline{H_A^-}$. In the other direction, if $\mathcal{C} \subset \overline{H_A^-}$, by the strict log-concavity of the determinant, the inequality $\det M < \det A$ holds on the whole cell. By Lemma 3.1, $\mathcal{C}$ is well rounded, and since A belongs to $\overline{\mathcal{C}}$, the unique maximum of the determinant on $\overline{\mathcal{C}}$ is attained at A.

Conversely, let $\mathcal{C}$ be a well-rounded cell without a weakly eutactic matrix. The unique matrix A at which the determinant attains its maximum on $\overline{\mathcal{C}}$ is weakly eutactic (because it is a maximum in its own cell) and is not eutactic: otherwise H_A^+ would contain $\mathcal{C}$, and this would imply $\det M > \det A$ on $\mathcal{C}$, a contradiction.

The last assertion is clear. □

We now establish some properties of the automorphism groups of cells. Recall that the automorphism group $\mathrm{Aut}(\mathcal{C})$ of a cell $\mathcal{C}$ is the stabilizer of $\mathcal{C}$ in $\mathrm{GL}_n(\mathbb{Z})$. We denote by $\mathrm{Aut}^+(\mathcal{C})$ the subgroup of $\mathrm{Aut}(\mathcal{C})$ consisting of positive automorphisms. For every $M \in \mathcal{C}$, $\mathrm{Aut}(M)$ is a subgroup of $\mathrm{Aut}(\mathcal{C})$, and is equal to the whole group $\mathrm{Aut}(\mathcal{C})$ if M is weakly eutactic ([Bt], Corollary 2.6). This accounts for the connection between Ash's original formula and the formulae stated in the introduction.

From now on, A stands for a weakly eutactic, non-eutactic matrix of minimum 1. Recall that $\mathcal{E}_A = \{\mathcal{C} \prec \mathcal{C}_A \mid \inf_{\mathcal{C}} \gamma = \gamma(A)\}$. We set $G = \mathrm{Aut}(A)$ and $G^+ = \mathrm{Aut}^+(A)$.

Lemma 3.3. *The group $G = \mathrm{Aut}(A)$ stabilizes $\mathcal{E}_A$, and every equivalence between two cells of $\mathcal{E}_A$ is the restriction of an automorphism of A. In particular, for every $\mathcal{C} \in \mathcal{E}_A$, $\mathrm{Aut}(\mathcal{C})$ can be identified with a subgroup of G.*

Proof. This is easily proved, using the fact that for every cell in $\mathcal{E}_A$, A is the *unique* matrix at which $\inf_{\mathcal{C}} \gamma$ is attained. □

The lemma above shows that in $\mathcal{E}_A$, two cells are equivalent under $\mathrm{GL}_n(\mathbb{Z})$ (or $\mathrm{SL}_n(\mathbb{Z})$) if and only if they belong to the same orbit under G (or G^+). Now for every $\mathcal{C}_0 \in \mathcal{E}_A$, we have

$$\frac{1}{|\mathrm{Aut}(\mathcal{C}_0)|} = \frac{1}{|\mathrm{Aut}(A)|} \sum_{\mathcal{C} \in \mathcal{E}_A,\, \mathcal{C} \sim \mathcal{C}_0} 1\,.$$

Thus Theorem 1.2 is equivalent to Theorem 3.4 below:

Theorem 3.4. *For every weakly eutactic, non-eutactic matrix A,*

$$\sum_{\mathcal{C} \in \mathcal{E}_A} (-1)^{\dim \mathcal{C}} = 0\,,$$

where $\mathcal{E}_A$ denotes the set of cells $\mathcal{C}$ such that $\inf_{\mathcal{C}} \gamma$ is attained at A.

The proof of Theorem 3.4 will rely on Euler-type formulae for convex polytopes. Traditionally one writes Euler's formula for a p-dimensional polytope P in the form

$$\sum_{i=0}^{p-1} (-1)^i n_i = 1 - (-1)^p$$

where n_i denotes the number of i-dimensional proper faces. The consideration of the improper two faces P and $\emptyset$, of dimensions p and -1

respectively, allows us to write down this formula in a form which no longer depends on p, namely

$$\sum_{i=-1}^{p} (-1)^i n_i = 0\,.$$

This latter form has the following generalization:

Theorem 3.5. *Let $\mathcal{F}_1$ and $\mathcal{F}_2 \supsetneqq \mathcal{F}_1$ be two faces of a given convex polytope P. Then*

$$\sum_{\mathcal{F}_1 \subset \mathcal{F} \subset \mathcal{F}_2} (-1)^{\dim \mathcal{F}} = 0\,.$$

Proof. This is Theorem 16.4 of [Br]. □

The theorem above has an evident counterpart for cones in vector spaces, simply replacing the empty face by the origin, as one immediately sees by considering sections by a transverse hyperplane. Using this device, we easily dispose of the case of semi-eutactic matrices.

Proof of Theorem 3.4 for semi-eutactic matrices. Proposition 2.4 and Lemma 2.1 show that if A is semi-eutactic but not eutactic, there is in H_A a unique maximal cell $\mathcal{C} \precneqq \mathcal{C}_A$, so that Theorem 3.5 applies directly, taking $\mathcal{F}_1 = \mathcal{D}_\mathcal{C}$ and $\mathcal{F}_2 = \mathcal{D}_A$ and using the correspondence of Lemma 2.1. □

The existence of unique maximal cell $\mathcal{C} \prec \mathcal{C}_A$ in $\overline{H_A^-}$ is no longer true for general weakly eutactic matrices. To handle this case, we first introduce an enlargement of the Voronoi domain.

Definition 3.6. *Let $\mathcal{D}'_A$ be the convex hull of the Voronoi domain $\mathcal{D}_A$ of A and the half-line $\mathbb{R}_{\geq 0}\, A^{-1}$.*

Since A is assumed to be weakly eutactic, A^{-1} belongs to the span of $\mathcal{D}_A$. Hence $\mathcal{D}_A$ and $\mathcal{D}'_A \supset \mathcal{D}_A$ have the same dimension, and $\mathcal{D}'_A = \mathcal{D}_A$ if and only if A is semi-eutactic. As above, the consideration of a transverse hyperplane through A^{-1} allows us to apply Theorem 3.5 to $\mathcal{D}'_A$.

In the remaining of this section, we assume that A is weakly eutactic, but not semi-eutactic. We first give a new characterization of the set $\mathcal{E}_A$.

Proposition 3.7. *Suppose that A is weakly eutactic, but not semi-eutactic, and let $\mathcal{C} \prec \mathcal{C}_A$. Then $\mathcal{C}$ is contained in $\overline{H_A^-}$ if and only if $\mathcal{D}_\mathcal{C}$ is not a face of $\mathcal{D}'_A$.*

Proof. Suppose first that $\mathcal{D}_{\mathcal{C}}$ is not a face of $\mathcal{D}'_A$. If $\mathcal{C} \not\subset \overline{H_A^-}$, there exists $M \in \mathcal{C} \cap H_A^+$. By Lemma 2.1, there exists a supporting hyperplane $\mathcal{H}$ of $\mathcal{D}_A$, orthogonal to $F = M - A$, and such that $\mathcal{H} \cap \mathcal{D}_A = \mathcal{D}_{\mathcal{C}}$. Since $M \in H_A^+$, we have $\langle F, A^{-1} \rangle > 0$. We then have $\mathcal{D}'_A \subset \overline{H_A^+}$, which shows that $\mathcal{H}$ supports $\mathcal{D}'_A$. Let $X' = \lambda A^{-1} + X \in \mathcal{D}'_A$ (with $\lambda \geq 0$ and $X \in \mathcal{D}_A$). Then $\langle X', F \rangle = \lambda \langle A^{-1}, F \rangle + \langle X, F \rangle$ (where $\langle X, F \rangle \geq 0$ and $\langle A^{-1}, F \rangle > 0$) vanishes if and only if $X' = X$ belongs to $\mathcal{D}_A \cap \mathcal{H} = \mathcal{D}_{\mathcal{C}}$. In other words, $\mathcal{D}'_A \cap \mathcal{H} = \mathcal{D}_{\mathcal{C}}$, a contradiction.

Conversely, suppose that $\mathcal{D}_{\mathcal{C}}$ is a face of $\mathcal{D}'_A$, and let $\mathcal{H}'$ be a supporting hyperplane for $\mathcal{D}'_A$ such that $\mathcal{D}'_A \cap \mathcal{H}' = \mathcal{D}_{\mathcal{C}}$. Let $F' \in \mathcal{H}'^{\perp}$ such that $\langle F', \mathcal{D}'_A \rangle \geq 0$. By Lemma 2.1, for $\varepsilon > 0$ small enough, $M = A + \varepsilon F'$ lies in $\mathcal{C}$. We have $\langle F', A^{-1} \rangle \geq 0$, and indeed $\langle F', A^{-1} \rangle > 0$ (otherwise, A^{-1} would belong to $\mathcal{D}'_A \cap \mathcal{H}' = \mathcal{D}_{\mathcal{C}}$, and A would be semi-eutactic), hence M belongs to H_A^+, and the cell is not included in $\overline{H_A^-}$. □

Proof of Theorem 3.4 for weakly eutactic, non-semi-eutactic matrices. The cone $\mathcal{D}'_A$ is the convex hull of the extreme edges of $\mathcal{D}_A$ and of the half-line $\mathbb{R}_{\geq 0} A^{-1}$. Hence its faces are either faces of $\mathcal{D}_A$ or contain A^{-1}. We must show that the sum

$$\Sigma = \sum_{\substack{\mathcal{F} \text{ face of } \mathcal{D}_A \\ \mathcal{F} \text{ not a face of } \mathcal{D}'_A}} (-1)^{\dim \mathcal{F}}$$

is zero. To this end, we introduce the sums

$$\Sigma' = \sum_{\substack{\mathcal{F} \text{ face of } \mathcal{D}_A \\ \mathcal{F} \text{ face of } \mathcal{D}'_A}} (-1)^{\dim \mathcal{F}} \quad \text{and} \quad \Sigma'' = \sum_{\substack{\mathcal{F} \text{ face of } \mathcal{D}'_A \\ A^{-1} \in \mathcal{F}}} (-1)^{\dim \mathcal{F}} .$$

Applying thrice Euler's generalized theorem3.5 to convenient cones and faces, we successively obtain $\Sigma'' = 0$, $\Sigma' + \Sigma'' = 0$, and $\Sigma + \Sigma' = 0$. □

To finish this section, we generalize to $\mathcal{D}'_A$ a well-known result of Voronoi, namely that all edges $\mathbb{R}_{\geq 0}\, x\,{}^t\!x$, $x \in S(A)$ are extreme for $\mathcal{D}_A$. We first prove a result about eutaxy relations:

Proposition 3.8. *In any eutaxy relation $A^{-1} = \sum_{x \in S(A)/\pm} a_x\, x\,{}^t\!x$, the set of $x \in S(A)$ with $a_x > 0$ spans $\mathbb{R}^n$.*

Proof. Let E be the span of $x \in S(A)$, $a_x > 0$, and let $y \in E^{\perp}$. We have

$${}^t\!y A^{-1} y = \sum_x a_x\, ({}^t\!y x)^2 = \sum_{a_x < 0} a_x\, ({}^t\!y x)^2 .$$

The left hand side is ≥ 0 while the right hand side is ≤ 0, hence both are zero, which implies $y = 0$ since A^{-1} is positive definite. This shows that $E = \mathbb{R}^n$. □

Proposition 3.9. *The extreme edges of $\mathcal{D}'_A$ are $\mathbb{R}_{\geq 0}\, A^{-1}$ and the $\mathbb{R}_{\geq 0}\, x\,{}^t\!x$, $x \in S(A)$.*

Proof. It is clear that every extreme edge of $\mathcal{D}'_A$ belongs to the set $\{\mathbb{R}_{\geq 0}\, A^{-1}\} \cup \{\mathbb{R}_{\geq 0}\, x\,{}^t\!x,\ x \in S(A)\}$. Conversely, since A is not semi-eutactic, $\mathbb{R}_{\geq 0}\, A^{-1}$ is an extreme edge. Let now $x \in S(A)$. If $\mathbb{R}_{\geq 0}\, x\,{}^t\!x$ were not extreme, there would exist a relation

$$x\,{}^t\!x = \lambda\, A^{-1} + \sum_{y \neq x} \lambda_y\, {}^t\!yy$$

with non-negative coefficients λ, λ_y. By Voronoi's theorem, λ is strictly positive, so that we have a eutaxy relation with exactly one positive coefficient (that of $x\,{}^t\!x$), which contradicts Proposition 3.8. □

4. Algorithmic remarks.

In the preceding two sections, the eutaxy property was studied using convexity. We now return to the notation introduced at the beginning of the introduction, considering eutaxy for a matrix A through the eutaxy coefficients a_x, $x \in S(A)/\pm$. A *perfection relation on S* is an equality $\sum_{x \in S/\pm} u_x\, x\,{}^t\!x = 0$. The set of perfection relations is a real vector space $\mathcal{P}_S$ of dimension $s - r$ (r denotes the perfection rank of S), that we also denote by $\mathcal{P}_A$ when $S = S(A)$. If A is weakly eutactic, the set $\mathcal{R}_A$ of eutaxy relations for A is an affine space over $\mathcal{P}_A$.

To carry out calculations, we use an arbitrary basis for $\mathcal{P}_A$, say, $\sum_{x \in S/\pm} u_x^{(i)}\, x\,{}^t\!x = 0$, $i = 1, \dots, s - r$. For short, we write these relations $\sum_{x \in S/\pm} U_x\, x\,{}^t\!x = 0$, where $U_x \in \mathbb{R}^{s-r}$ is the column-matrix with components $u_x^{(i)}$.

We denote by S_0 the set of vectors $x \in S$ such that $U_x = 0$. When A is weakly eutactic, the eutaxy coefficients a_x, $x \in S_0$ do not depend on the choice of a eutaxy relation. Thus the conditions $a_x < 0$, $a_x = 0$, $a_x > 0$ respectively define a partition $S_0 = S_0^- \cup S_0^0 \cup S_0^+$ of S_0.

We intend to return somewhere else to the algorithmic aspects of extractions of cells and just state below without a proof our main result, which we shall use to construct cells of co-rank one below a given cell. As we shall see in Proposition 4.2, these cells play a crucial rôle in the determination of the set $\mathcal{E}_A$ (defined above Theorem 1.2).

Theorem 4.1. *Let A, r, s, S_0 and $\{U_x,\ x \in S\} \subset \mathbb{R}^{s-r}$ be as above, and let $x_1 \dots, x_k \in S$ ($k \geq 1$) be k minimal vectors. The following conditions are equivalent:*

(1) *$S' = S \smallsetminus \{\pm x_1, \dots, \pm x_k\}$ is the set of minimal vectors for a cell $\mathcal{C}'$ of perfection rank $r - 1$.*

(2) *The set* $\{U_{x_1}, \dots, U_{x_k}\}$ *has rank* $k-1$ *and there exists a relation* $\sum_{i=1}^{k} \rho_i U_{x_i} = 0$ *with strictly positive coefficients* ρ_i.

Moreover, for any $x \in S$*, there exist* $k-1 \geq 0$ *vectors* $x_2, \dots, x_k$ *such that* $x_1 = x, x_2, \dots, x_k$ *satisfy the conditions above, and* k *is equal to* 1 *if and only if* $x \in S_0$. □

The subset S_0 of S plays a special rôle in various respects. First Theorem 3.4 has an easy proof for any weakly eutactic, non-eutactic matrices A with $S_0^- \cup S_0^0 \neq \emptyset$: indeed, given $x_0 \in S_0^- \cup S_0^0$, cells below $\mathcal{C}_A$ belonging to $\overline{H_A^-}$ occur in pairs $(\mathcal{C}, \mathcal{C}')$ with $S(\mathcal{C}) = S(\mathcal{C}') \cup \{\pm x_0\}$, which cancel in Theorem 3.4.

Actually, given $\mathcal{C} \subset \overline{H_A^-}$, for any set $x_1, \dots, x_\ell \in (S_0^- \cup S_0^0) \smallsetminus S(\mathcal{C})$, the cell $\mathcal{C}' \prec \mathcal{C}$ with $S(\mathcal{C}') = S(\mathcal{C}) \smallsetminus \{\pm x_1, \dots, \pm x_\ell\}$ belongs to $\overline{H_A^-}$. In particular, if there exists a eutaxy relation whose coefficients a_x are strictly positive for all $x \in S \smallsetminus S_0$, then there is a unique maximal cell $\mathcal{C}' \prec \mathcal{C}_A$, namely the cell with minimal vectors $S \smallsetminus (S_0^- \cup S_0^0)$. In general, such a cell does not exist when A is not semi-eutactic; see the end of this section.

Moreover, it is an experimental fact that very often, there exists a eutaxy relation $\mathcal{R}$ for which $S^-(\mathcal{R}) = S_0^-$ and $S^0(\mathcal{R}) = S_0^0$. This is true for $n \leq 5$ (because all cells with $s > r$ contain a eutactic matrix whereas $S = S_0$ if $s = r$), and for all perfect matrices with $n \leq 7$.

However semi-eutactic matrices with $T \supsetneqq S_0$ (see Proposition 2.4) exist in dimension 6, where an example is provided by a matrix with $s = 21$, $r = 17$ and equal non-zero eutaxy coefficients (so that the corresponding minimal vectors constitute a spherical 3-design), found by Elbaz-Vincent, Gangl and Soulé; see the talk by Elbaz-Vincent in [Ob].

Also, a perfect, semi-eutactic matrix with $T \supsetneqq S_0$ is provided by a *Laïhem lattice* (a perfect 8-dimensional lattice having a perfect 7-dimensional section with the same minimum). Gram matrices for these lattices are displayed in [Bt-M] (use the command lh(i)), and Jaquet has listed those which are eutactic or semi-eutactic, non-eutactic (all are weakly eutactic since they are perfect). (Jaquet's unpublished results have been recently confirmed by Cordian Riener ([Ri]), who has tested for eutaxy all perfect 8-dimensional lattices — Dutour, Schürmann and Vallentin have recently announced that the list of perfect, 8-dimensional lattices displayed in[Bt-M] is indeed complete.) There are exactly 21 semi-eutactic, non-eutactic Laïhem lattices, and for 20 of them, there exists a eutaxy relation for which $S^0 = S_0^0$. For the remaining lattice, namely lh(958), with $s = 40$, S_0 reduces to S_0^+, and there exist minimal vectors y, z with $U_z + U_y = 0$ and a eutaxy relation with coefficients $a_y = a_z = 0$. In this case, $T = \{\pm y, \pm z\}$, and $S \smallsetminus T$ is the set of minimal vectors of the unique cell $\mathcal{C} \precneqq \mathcal{C}_A$ contained in H_A.

Various examples of perfect, weakly eutactic, non-semi-eutactic matrices A such that H_A^- contains cells $\mathcal{C}$ with $S(A) \smallsetminus S(\mathcal{C}) \not\subset S_0$ have been found among the Laïhem lattices. For instance, there are 154 Laïhem lattices with $s = 38$ among which 5 (e.g., lh(183)) provide such a cell.

Proposition 4.2. *Let A be a weakly eutactic matrix and let $\mathcal{C} \precneqq \mathcal{C}_A$. Denote by $\mathcal{C}_1, \dots, \mathcal{C}_k$ the cells of rank $r-1$ such that $\mathcal{C} \prec \mathcal{C}_i \prec \mathcal{C}_A$. Then*

$$\mathcal{C} \subset \overline{H_A^-} \Longleftrightarrow \forall\, i,\, \mathcal{C}_i \subset \overline{H_A^-}\,.$$

Proof. If $\mathcal{C} \subset \overline{H_A^-}$, then $\mathcal{C}_i \subset \overline{\mathcal{C}}$ is also contained in $\overline{H_A^-}$.

Let us now prove the converse. For $i = 1, \dots, k$ let $F_i \in V_A$ be "facet-matrices" for $\mathcal{C}_i$ (see Lemma 2.1; we have $\langle F_i, x\,{}^t\!x \rangle = 0$ for $x \in S(\mathcal{C}_i)$ and $\langle F_i, x\,{}^t\!x \rangle > 0$ for $x \in S(A) \smallsetminus S(\mathcal{C}_i)$). Then for any $M \in \mathcal{C}$ there exist, as we now prove, $V \in V_A^\perp$ and $\alpha_i \geq 0$, $i = 1, \dots, k$ such that

$$M = A + V + \sum_{i=1}^{k} \alpha_i F_i\,. \qquad (**)$$

Indeed, let $\mathcal{F}_i = \mathcal{D}_{\mathcal{C}_i}$ (resp. $\mathcal{F} = \mathcal{D}_{\mathcal{C}}$) be the facets of $\mathcal{D}_A$ (resp. the face of $\mathcal{D}_A$) associated with the cells $\mathcal{C}_i$ (resp. with $\mathcal{C}$); thus the $\mathcal{F}_i$ are the facets of $\mathcal{D}_A$ containing $\mathcal{F}$. Denote by F the orthogonal projection to V_A of $M - A$, so that $F^\perp$ is a supporting hyperplane of $\mathcal{D}_A$ for $\mathcal{F}$; see Lemma 2.1. We must prove that F is a positive linear combination of the F_i. Let $C = \{v \in V_A \mid \forall\, i,\, \langle v, F_i \rangle \geq 0\}$ and let $v \in C$. Replacing v by a vector w of the form $w = h\,v + (1-h)\,u$ for some non-zero $u \in \mathcal{F}$ and a small enough $h > 0$, we may assume that v belongs to $\mathcal{D}_A$. We then have $\langle v, F \rangle \geq 0$, so that the cone defined by the inequalities $\langle v, F_i \rangle \geq 0$ and $\langle v, -F \rangle \geq 0$ is contained in the hyperplane $F^\perp$. By Voronoi's "fundamental principle" used in the proof of Proposition 2.2, $-F$ and the F_i are positively dependent. Since the F_i are not positively dependent (the cone C has maximal dimension by [Br], Corollary 11.7), F is a positive combination of the F_i, say $\sum_i \alpha_i\, F_i$, and $(**)$ holds taking $V = (M - A) - F$. By $(**)$, since V belongs to $V_A^\perp$ and A^{-1} to V_A, we have $\langle M - A, A^{-1} \rangle = \langle F, A^{-1} \rangle \leq 0$. $\square$

We now make more precise the case when $s = r + 1$. (As above, r stands for the perfection rank of A.) There is (up to a scaling factor) a unique perfection relation $\sum_{x \in S(A)/\pm} u_x\, x\,{}^t\!x = 0$. We also chose a eutaxy relation $A^{-1} = \sum_{x \in S(A)/\pm} a_x\, x\,{}^t\!x$. The set $S(A)$ can be written as a disjoint union $S(A) = S_0 \cup P^+ \cup P^-$ where $P^+ = \{x \in S(A) \mid u_x > 0\}$ and $P^- = \{x \in S(A) \mid u_x < 0\}$. (Note that only the unordered set $\{P^+, P^-\}$ is intrinsic.)

Proposition 4.3. *Let A with $s(A) = r + 1$. A cell $\mathcal{C} \prec \mathcal{C}_A$ of rank $r - 1$ is of one of the following two types:*

(1) $S(\mathcal{C}) = S(A) \smallsetminus \{\pm z\}$, $z \in S_0$.
(2) $S(\mathcal{C}) = S(A) \smallsetminus \{\pm x, \pm y\}$, $x \in P^+$, $y \in P^-$.

The cell $\mathcal{C}$ is contained in H_A^- (resp. in H_A) if and only if $z \in S_0^-$ (resp. $z \in S_0^0$) in case (1), and $\frac{a_x}{u_x} < \frac{a_y}{u_y}$ (resp. $\frac{a_x}{u_x} = \frac{a_y}{u_y}$) in case (2).

Proof. The description of the possible cells $\mathcal{C}$ results directly from Theorem 4.1. Let F such that $\langle F, x^t x \rangle$ is zero if $x \in S(\mathcal{C})$ and strictly positive if $x \in S(A) \smallsetminus S(\mathcal{C})$. We have

$$\langle F, A^{-1} \rangle = \sum_{\pm x \in S(A) \smallsetminus S(\mathcal{C})} a_x \, \langle F, x^t x \rangle \quad \text{and} \quad \sum_{\pm x \in S(A) \smallsetminus S(\mathcal{C})} u_x \, \langle F, x^t x \rangle = 0 \,.$$

Recall (Lemma 2.1) that the position of $\mathcal{C}$ with respect to H_A depends on the sign of $\langle F, A^{-1} \rangle$. In case (1), $\langle F, A^{-1} \rangle$ has the sign of a_z. In case (2), we have

$$\langle F, A^{-1} \rangle = (-u_y \, \langle F, y^t y \rangle) \, (\frac{a_x}{u_x} - \frac{a_y}{u_y})$$

so that $\langle F, A^{-1} \rangle$ has the sign of $\frac{a_x}{u_x} - \frac{a_y}{u_y}$. □

Remark 4.4. Removing from $S(A)$ vectors of S_0 preserves the eutaxy relation $\sum_{x \in P^+ \cup P^-} u_x x^t x = 0$, while removing one pair $(\pm x, \pm y)$, $(x, y) \in P^+ \times P^-$ destroys it. Thus iterating Construction 4.2 we obtain two types of cells $\mathcal{C} \prec \mathcal{C}_A$ of perfection rank $r_{\mathcal{C}} = r - k$, $k \geq 1$: those obtained by removing from $S(A)$ k pairs $\pm z$ of vectors of S_0, and those obtained by removing $k + 1$ pairs $\pm x$, with at least one of them in P^+ and one of them in P^-.

We now give an example for which H_A^- contains two maximal cells $\mathcal{C} \prec \mathcal{C}_A$. We take for A a Gram matrix for the lattice nap(4118) (see [Bt-M] for the definition of the Napias lattices). Here, we have $r = \frac{8(8+1)}{2} = 36$ and $s = 37$.

In the following we denote by $[R]$ the cell below $\mathcal{C}_A$ obtained by removing from $S(A)$ the vectors of R, and we shall use this notation to describe the set $\mathcal{E}_A = \{\mathcal{C} \prec \mathcal{C}_A \text{ and } \mathcal{C} \subset \overline{H_A^-}\}$. The perfection relation of A reads $\sum_{x \in P^+} x^t x - \sum_{x \in P^-} x^t x = 0$, with $|P^+| = |P^-| = 8$ (we count the pairs $\pm x$). With the notation of Proposition 4.3 the set S_0^0 is empty, S_0^- consists of a single pair $\pm z_0$, and the differences $\frac{a_x}{u_x} - \frac{a_y}{u_y}$ for

$(x,y) \in P^+ \times P^-$ are strictly positive except for threepairs $(\pm x_1, \pm y_1)$, $(\pm x_2, \pm y_2)$ and $(\pm x_1, \pm y_2)$, with respective values -44, -44 and -444. By Proposition 4.3, we see that among the 85 cells of rank 35 below C_A, only four are contained in $\overline{H_A^-}$ (actually, in H_A^-): the cell $[\pm z_0]$ of type (1) and the cells $[\pm x_1, \pm y_1]$, $[\pm x_2, \pm y_2]$ and $[\pm x_1, \pm y_2]$ of type (2). Let $[R] \prec \mathcal{C}_A$ be a cell of rank ≤ 34. The cells of rank 35 above $[R]$ are the $[\pm z]$ and the $[\pm x, \pm y]$ with $x, y, z \in R$, $x \in P^+$, $y \in P^-$ and $z \in S_0$. Thus by Proposition 4.2 the cell $[R]$ belongs to $\mathcal{E}_A$ if and only if R is included in one of the sets $\{\pm z_0, \pm x_1, \pm x_2, \pm y_2\}$ or $\{\pm z_0, \pm x_1, \pm y_1, \pm y_2\}$. [Indeed R must not contain both x_2 and y_1, since the cell $[\pm x_2, \pm y_1]$ is contained in H_A^+.] Taking into account the general form of the cells below $\mathcal{C}_A$ (see Remark 4.4), we can complete the enumeration of the cells belonging to $\mathcal{E}_A$. There are five cells of rank 34, namely $[\pm z_0, \pm x_1, \pm y_2]$, $[\pm z_0, \pm x_2, \pm y_2]$, $[\pm x_1, \pm x_2, \pm y_2]$, $[\pm z_0, \pm x_1, \pm y_1]$ and $[\pm x_1, \pm y_1, \pm y_2]$; two cells of rank 33, namely $[\pm z_0, \pm x_1, \pm x_2, \pm y_2]$ and $[\pm z_0, \pm x_1, \pm y_1, \pm y_2]$; and no cell of rank ≤ 32. So, the set $\mathcal{E}_A$ contains 1+4+5+2 cells, and Theorem 3.4 enables us to check this enumeration:

$$1 - 4 + 5 - 2 = 0\,.$$

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